QUESTION 1

Let:

$$
\begin{aligned}
A &= \lim_{x\to 4} \frac{x^3-64}{\sqrt{x}-2} \\
B &= \lim_{x\to\infty} \left(\sqrt{x^2-46x+2020}-\sqrt{x^2+46x+2021}\right) \\
C &= \lim_{x\to 0} \frac{(x+4)^{\frac{3}{2}}+e^x-9}{x} \\
D &= \lim_{x\to\infty} x\ln\left(\frac{27+27x}{x^3}+\frac{9+x}{x}\right)
\end{aligned}
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Find $A+B+C+D$.

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QUESTION 2

Consider the curve $y/x = \ln(x/y)$:

A = The slope of the normal line to the graph at $x = 21$

B = The area bound by the graph, the x-axis, and the lines $x = 1$ and $x = 21$

Find AB.

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Find AB.

QUESTION 3

Newton approximates the root of $f(x) = x^2$ by using his famous method (Newton's method) with n iterations and an initial guess of $x_0 = 80$. Let A be the minimum value of n for which his approximation is less than 0.01 from the actual root.

Euler is bad at squaring integers, so he decides to approximate n^2 by using his famous method (Euler's method) with n equal-length steps on the function $f(x) = x^2$, starting at $x_0 = 0$ and $f(x_0) = 0$. Let B be the minimum integer value of n for which the magnitude of the percent error in his approximation is less than 1%.

Euler is also bad at cubing integers, so he decides to enlist the help of his friend Riemann. Riemann approximates $\int_0^n 3x^2 dx$ using a trapezoidal Riemann sum with n equally-spaced intervals. Let C be the minimum value of n for which the percent error in his approximation is less than 1%.

Find $A + B + C$.

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Find $A + B + C$.

QUESTION 4

A particle X travels along the x-axis with position dictated by the following equation:

$$x(t) = 4t^3 - 28t$$

A second particle Y travels along the y-axis with velocity $y'(t) = x(t)$, and $y(0) = 25$.

Let:

$$\begin{aligned} A &= \text{the maximum speed of particle } X \text{ on the interval } t \in [-1, 2] \\ B &= \text{the total distance that particle } Y \text{ travels from } t = 1 \text{ to } t = 3 \\ C &= \text{the maximum distance between particles } X \text{ and } Y \text{ on the interval } t \in [-2, 2] \end{aligned}$$

Find $A + B + C^2$.

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Find $A + B + C^2$.

QUESTION 5

There exists a function $f(x)$ such that $f'(x) = 8-12x+2f(x)-3xf(x)$. The graph of $y = f(x)$ has a horizontal asymptote at $y = A$.

There exists a second function $g(x)$ such that $\sum_{i=0}^{\infty} g^{(i)}(x) = x^4 + 16x$. If $g(x)$ is a polynomial of degree 4, let B be the product of its *distinct* roots.

Rohan likes throwing perfectly spherical snowballs, but would like to throw them before they melt in the Florida heat. He knows that the volume of the snowball decreases proportionally to its current surface area, and that it always remains spherical as it melts. If it takes 3 minutes for the volume of the snowball to decrease to an eighth of its original volume, let C equal the amount of time, in minutes, it will take for the remaining snowball to melt completely.

Find ABC.

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Find ABC.

QUESTION 6

Farzan and Dylan are competing with numbers. Farzan randomly chooses a positive number f less than 8. Dylan randomly chooses a number d between 1 and 10. Let A be the probability that $d^2 + 40 > 14d - f$

Alex, Vishnav, and Akash each randomly and independently choose positive numbers less than 1. Let B be the probability that the sum of the square of Alex's number, four times the square of Vishnav's number, and four times the square of Akash's number, is less than 4.

Find $81A + 96B$.

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Find $81A + 96B$.

QUESTION 7

Josh, Rayyan, and Nihar all start at the same location and begin moving at the same time. Each moves in a direction perpendicular to the directions that the other two are moving, Josh at 10 miles per hour, Nihar at 20 miles per hour, and Rayyan at 30 miles per hour. After 3 hours, let A equal the rate at which the area of the triangle with vertices at the 3 friends' locations is increasing (in square miles per hour).

Karthik enjoys playing with sand and is dumping sand into a pile that takes the shape of a cone whose base diameter and height are always equivalent. If he is dumping sand into the pile at a rate of 30 cm^3 per second, let B equal the rate of increase of the height of the pile, in cm per second, when the pile is 10 cm high.

In Tallahassee, there is a certain streetlight that is 15 feet tall. Tanvi, who is 6 feet tall (wow!), walks directly away from the pole at 5 feet per second. Assuming this streetlight is the only light source, let C equal the speed of the tip of Tanvi's shadow, in feet per second, when she is 40 feet from the pole.

Find $\dfrac{A}{BC}$.

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Find $\dfrac{A}{BC}$.

QUESTION 8

Let:

$$\begin{aligned} A &= \int_0^1 \ln(x)dx \\ B &= \int_0^3 \frac{1}{\sqrt{x}}dx \\ C &= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{(e-1)\ln\left(\frac{ei-i+n}{n}\right)}{n} \\ D &= \lim_{n\to\infty}\sum_{i=1}^{3n} \sqrt{\frac{n^{1/2}+2i^{1/2}+in^{-1/2}}{n^{5/2}}} \end{aligned}$$

Find $A - B + C + D$.

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Find $A - B + C + D$.

QUESTION 9

Let:

A = the arc length of the portion of the curve $r = 6\cos(\theta) - 3\sin(\theta)$ for $0 \leq \theta \leq \pi$

B = the surface area of the object obtained by rotating $y = 2x + 6$ from $1 \leq x \leq 4$ about the y-axis

C = the volume of a wedge cut from a cylinder of radius $2\sqrt{3}$ (and arbitrarily large height) by a plane that intersects the cylinder at its base at an angle of $\frac{\pi}{6}$ and goes through the diameter of the base of the cylinder

D = the volume of the intersection of two spheres of radius 2, situated such that the center of each sphere lies on the surface of the other sphere

Find $(\frac{B}{A} - C)(D)$.

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Find $(\frac{B}{A} - C)(D)$.

QUESTION 10

Vishnav is studying hard for Sequences and Series. Let:

$$A = 1 \text{ if the following series converges or 2 if it diverges: } \sum_{n=1}^{\infty} \frac{n+5}{n\sqrt{n+3}}$$

$$B = 3 \text{ if the following series converges or 4 if it diverges: } \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 6^n}$$

$$C = \text{the radius of convergence for the following series, or -1 if it always diverges: } \sum_{n=0}^{\infty} \frac{n^3 x^{3n}}{n^4+1}$$

Now let $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$ and $\eta(s) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^s}$. Find $\frac{\eta(A+B+C)}{\zeta(A+B+C)}$

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QUESTION 11

A particle in a mysterious force field follows the differential equations $x''(t) = -4x$ and $y''(t) = -4y$. At $t = 0$, it is at the point $(4, 0)$ with initial velocity of v_0 in the $+y$ direction. Let:

$$\begin{aligned} A &= \text{The value of } |v_0| \text{ for which the particle follows a circular path} \\ B &= \text{The number of times the particle crosses the } y\text{-axis from } t = 0 \text{ to } t = 30\text{, if } v_0 = 2021 \end{aligned}$$

Now consider a scenario in which $v_0 = 12$ and the force on the particle suddenly ceases at $t = \pi/6$, causing it to continue moving in a straight line. Let C be the y-coordinate of the particle at the next time it crosses the y-axis.

Find $(A + B)(C)$

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QUESTION 12

Rayyan constructs a list of points on the curve $y = x^3$, labeled P_0, P_1, P_2... The tangent line to $y = x^3$ at P_i intersects $y = x^3$ again at P_{i-1}, for all positive integer values of i.

$$\begin{aligned} A_1 &= \text{The area bound by line segment } P_0P_1 \text{ and } y = x^3 \\ A_2 &= \text{The area bound by line segment } P_1P_2 \text{ and } y = x^3 \\ A_3 &= \text{The area bound by line segment } P_2P_3 \text{ and } y = x^3 \\ &\ldots \\ A_i &= \text{The area bound by line segment } P_{i-1}P_i \text{ and } y = x^3 \end{aligned}$$

Given that the x-coordinate of P_0 is $2\sqrt{5}$, find $\sum_{i=1}^{\infty} A_i$

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A &= \int_0^{\pi} \left(e^x \sin(x)\right) dx \\
B &= \text{The sum of the digits of } \int_0^{98} \left(x^3 + 6x^2 + 12x + 18\right) dx \\
C &= \int_0^{\frac{\pi}{2}} \left(30 \sin^5(x)\right) dx \\
D &= \int_0^{\pi} \left(e^x \cos(x)\right) dx
\end{aligned}
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Find $A + B + C - D$.

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Find $A + B + C - D$.

QUESTION 14

Given:

$$\begin{aligned} f(x) &= (\log_2 x)^5 + \cos\left(\frac{\pi}{x-4}\right) \\ g(x) &= \int_0^{x^3} \left(\cos(\sqrt[3]{t}) + \sin(\sqrt[3]{t})\right) dt \\ h(x) &= \frac{|x^2 - 6x + 8|}{x+7} \end{aligned}$$

Let:

$$\begin{aligned} A &= f'(5) \\ B &= g'(\pi) \\ C &= g''(\pi) \\ D &= h'(3) \end{aligned}$$

Find $\dfrac{(\log_2 5)^5}{A} + \dfrac{B-C}{\pi} - \dfrac{1}{D}$.

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