1. We can sketch out a right triangle with one angle $x/2$, the opposite leg u, and the adjacent leg 1, allowing for a tangent of u. Then the length of the hypotenuse is $\sqrt{1+u^2}$, so $\sin(x/2) = \frac{u}{\sqrt{1+u^2}}$ and $\cos(x/2) = \frac{1}{\sqrt{1+u^2}}$. Using double-angle formulas, $\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2u}{1+u^2}$ and $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = \frac{1-u^2}{1+u^2}$. Finally, $\frac{du}{dx} = \frac{1}{2}\sec^2(x/2)$, so $\frac{dx}{du} = 2\cos^2(x/2) = \frac{2}{1+u^2}$. Combining them we get $\frac{(1-u^2)-2+2u}{1+u^2}$ which simplifies to

$$\boxed{\frac{-(u-1)^2}{u^2+1}}$$

2. First we find the intersections of the two functions at $x^{4n+1} = x^{4n+3}$, meaning $x = 0$ or $x^2 = 1 \to x = -1, 1$. Since these functions are odd due to the odd powers, if one is above the other for positive x values, it will be below for negative x values. Thus, if we integrate from -1 to 1 we will get an answer of 0, so instead we must integrate from 0 to 1 and double the area. $A_n = 2\int_0^1 (x^{4n+1} - x^{4n+3})dx = \dfrac{2}{4n+2} - \dfrac{2}{4n+4}$. Thus the sum of all A_n from $n = 0$ to infinity is $2(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}\dots) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\dots = \boxed{\ln 2}$ using a well-known result, which can be derived by evaluating the Taylor series for $\ln(1+x)$ at $x = 1$.

3. Tanvi's tangent is the tangent to $y = \tan(x)$ at $x = a$. The slope is $\sec^2 a$ and the point $(a, \tan(a))$ is on the line, so we can use point-slope form, giving the line $y - \tan(a) = \sec^2 a(x-a)$. Since everything is symmetric about the y-axis, the two tangent lines intersect on the y-axis, so the y-coordinate of intersection is equal to the y-intercept of the line. Plugging in $x = 0$ and $y = -a$, we get $-a = \tan(a) - a\sec^2 a$, $a\sec^2 a - a = \tan(a)$, and using the trig identity $\sec^2 a - 1 = \tan^2 a$ we find that $a\tan^2 a = \tan(a)$. Since $\tan(a) \neq 0$, we can divide to find $a\tan(a) = 1$. There is no way to find the actual value of a from this, but it doesn't matter! The triangle with vertices at Tanusri, Tanvi, and the origin has base $2a$ and height $\tan(a)$, so its area is $a\tan(a) = \boxed{1}$.

4. According to half-angle formula, $\cos(x/2) = \sqrt{\frac{1+\cos(x)}{2}}$, so $2\cos(x/2) = \sqrt{2+2\cos(x)}$ and thus $f(x) = 10\cos(x/2)$. We can factor out the 10 and multiply it back in later. As always with sine and cosine, the sequence of derivatives will cycle through $\cos(x), -\sin(x), -\cos(x), \sin(x), \cos(x)\dots$. By chain rule, each derivative will also multiply the function by 1/2. Thus we can decompose the series into two infinite geometric series - one for cosine and one for sine. The cosine series goes $\cos(x/2), (-1/4)\cos(x/2), (1/16)\cos(x/2)\dots$, so it has a common ratio of $-1/4$. Using the normal $a/(1-r)$ geometric series formula and the fact that $\cos(\pi/6) = \sqrt{3}/2$, the sum of the cosine series is $\frac{2\sqrt{3}}{5}$. The sine series is almost exactly the same, except we must note that it starts with $(-1/2)\sin(x/2)$ and that $\sin(\pi/6) = 1/2$, giving a sum of $-\frac{1}{5}$. Combining these two and multiplying by 10, we get $S = \boxed{4\sqrt{3}-2}$.

5. We know that when $x = 0$, $x^a = 0$ and $a^x = 1$. We also know that as x grows to infinity, a^x will grow faster than x^a. Thus, in the first quadrant a^x "starts" and "ends" above x^a. However, we know that the two must intersect at at least one point: the trivial case $x = a$. The fewest first-quadrant intersections possible should then be exactly 1. For the two curves to intersect only once, with a^x always greater than or equal to x^a, they must be tangent at their point of intersection. If they were not tangent, they would have different slopes, and thus would cross, meaning they would intersect at another point as well. So, this boils down to finding the value of a for which $f(x) = x^a$

and $g(x) = a^x$ are tangent at $x = a$. $f'(x) = ax^{a-1}$ so $f'(a) = a^a$. $g'(x) = a^x \ln(a)$ so $g'(a) = a^a \ln(a)$. Therefore $a^a = a^a \ln(a)$ so $\ln(a) = 1$ and $a = e \approx 2.72$. $\lfloor 10e \rfloor \approx \lfloor 27.2 \rfloor = 27$, and the sum of the digits is $\boxed{9}$.

6. For any positive x, $\arctan(x)$ will be between 0 and $\pi/2$, so in $\tan(\arctan(x))$, arctangent produces a positive angle and tangent finds the tangent of that angle - in other words, it cancels out to become x. Thus the integral of the first function is just $(1/2)x^2 = \frac{9\pi^2}{8}$ It's tempting to think the same will be true for $\arctan(\tan(x))$, but we must remember traditional ranges! Since arctangent can only produce angles between $-\pi/2$ and $\pi/2$, if $x > \pi/2$ then $\arctan(\tan(x)) = \arctan(\tan(x - \pi) = x - \pi$. Thus the second function is piecewise with $y = x$ from 0 to $\pi/2$ and $y = x - \pi$ from $\pi/2$ to $3\pi/2$. Integrating both or just sketching the triangles gives an area of $\frac{\pi^2}{8}$. The difference between the two is $\boxed{\pi^2}$.

7. The slope of the tangent line to the curve at $(a, \sqrt{3-a})$ is $y' = -\frac{1}{2\sqrt{3-a}}$. The slope of the normal line is then $m = 2\sqrt{3-a}$. Using point-slope form, the equation of the line is $y - \sqrt{3-a} = 2\sqrt{3-a}(x - a)$, so the y-intercept occurs where $x = 0$: $y = (1 - 2a)\sqrt{3-a}$. To minimize the y-intercept, we take the derivative: $y' = -2\sqrt{3-a} + \frac{2a-1}{2\sqrt{3-a}} = \frac{6a-13}{2\sqrt{3-a}}$. The derivative switches from negative to positive at only one point, $a = 13/6$, so this must be the x-coordinate of a relative minimum. The corresponding y-intercept is then $(1 - 13/3)\sqrt{5/6} = \boxed{\dfrac{-5\sqrt{30}}{9}}$

8. First we solve for c in the MVT for derivatives. The derivative at c is nac^{n-1}, and the average rate of change across the interval is $\dfrac{ab^n - 0}{b - 0} = ab^{n-1}$. Simplifying, $c_D = \dfrac{b}{n^{1/(n-1)}}$.

In MVT for integrals, $\displaystyle\int_0^b ax^n dx = \frac{ab^{n+1}}{n+1}$. This means the average value on the interval is $ab^n/(n+1)$, which is equal to ac^n - solving, we get $c_I = \dfrac{b}{(n+1)^{1/n}}$.

The ratio c_I/c_D is thus equal to $\dfrac{n^{1/(n-1)}}{(n+1)^{1/n}}$. However, as n approaches infinity, $n - 1 = n = n + 1$, so the fraction simplifies to $\dfrac{n^{1/n}}{n^{1/n}} = \boxed{1}$.

9. Completing the square shows that the curve is $(x - 5)^2 + (y - 6)^2 = 4$, a circle centered at $(5, 6)$ with radius 2. The line of Shrung's slice passes through $(5, 6)$ because $16 - 2(5) = 6$, so it cuts along a diameter of the circle. By symmetry, the midpoint of the two semicircle centroids must be the center of the original circle, so $(x_1 + x_2)/2 = 5$ and $(y_1 + y_2)/2 = 6$. This gives us the sum of the coordinates, but for the last portion of the answer we must find the distance between them, which by symmetry is twice the distance from each centroid to the diameter. To find the distance from each centroid to the line, we essentially must find the distance from the centroid of a semicircle with radius R to its diameter. We can do this using integration. If we assume that the diameter of the semicircle lies on the y-axis and that the figure lies to the right of the y-axis, the COM calculation becomes $\displaystyle\frac{1}{A}\int_0^R xy dx = \frac{2}{\pi R^2}\int_0^R 2x\sqrt{R^2 - x^2} dx$. Using a standard u-substitution of $u = R^2 - x^2$, we compute the integral as $\displaystyle\frac{2}{\pi R^2}\int_0^{R^2} \sqrt{u} du = \frac{2}{\pi R^2} \cdot \frac{2}{3}(R^3) = \frac{4R}{3\pi}$. Plugging in $R = 2$, we find that the COM is $8/3\pi$ from the diameter, so the

total distance between the two COMs is $16/3\pi$. Multiplying by 6π and adding to the rest we get $32+10+12 = \boxed{54}$.

10. Using self-similarity, this simplifies to $y = e^{xy}$. Then taking the natural log of both sides we rearrange to get $x = \frac{\ln y}{y}$. The maximum value of x occurs when the derivative of the right side is 0: $x' = \frac{1-\ln(y)}{y^2}$. When $y < e$, the derivative is positive, and when $y > e$, the derivative is negative, so when $y = e$ then x is maximized. At $y = e$ the corresponding value of x is then $\boxed{\frac{1}{e}}$.

11. This volume is easiest to see by recognizing that $x = \frac{\ln y}{y}$ is the inverse function of $y = \frac{\ln x}{x}$. Therefore the region bound by the original and the y-axis is equal to the region bound by the inverse and the x-axis (they are symmetric across $y = x$). So we can just use disk method to find the volume when $y = \frac{\ln x}{x}$ is revolved around the x-axis. Sketching the curve, we see that it crosses the x-axis at $x = 1$ then asymptotes to 0 as x goes to infinity. Then the integral is $\pi \int_1^\infty \frac{(\ln x)^2}{x^2} dx$. To solve the integral, we can substitute $u = \ln x$ which gives $I = \int_0^\infty u^2 e^{-u} du$. Using integration by parts once, we find that $I = -u^2e^{-u} + 2\int ue^{-u}du$. Then using it again on the remaining integral, we get $I = -u^2e^{-u} + 2(-ue^{-u} - e^{-u})$. Plugging in the bounds, we find that $I = (0) - (-2) = 2$. Lastly, remember that the actual volume is $I\pi = \boxed{2\pi}$.

12. A real number is equal to the sum of the greatest integer less than it and the decimal part of the number - $x = \lfloor x \rfloor + \{x\}$. So the integrand expands to $\lfloor x \rfloor \sqrt{\{x\}} + \{x\}^{3/2}$. The floor portion can be converted to a summation: $15 \left(\sum_{i=0}^{19} i\right) \left(\int_0^1 \sqrt{x} dx\right) = 15\frac{(19)(20)}{2}(2/3) = 1900$ using the formula for sum of integers from 1 to n. The other portion is $15 \int_0^{20} \{x\}^{3/2} dx = 300 \int_0^1 x^{3/2} dx = 120(1^{5/2}) = 120$. Adding them together we get $\boxed{2020}$.

13. Given a starting value x_0, Newton's method progresses by finding the x-intercept of the tangent line to $f(x)$ at x_0. The equation of the line is $y - f(x_0) = f'(x_0)(x - x_0)$ so the x-intercept occurs when $y = 0$: $-f(x_0) = f'(x_0)(x_1 - x_0) \rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. Then, since we know $f(x_0) = (x_0)^{2021}$ and $f'(x_0) = 2021(x_0)^{2020}$, $x_1 = x_0 - \frac{1}{2021}x_0 = x_0(1 - \frac{1}{2021})$. By the same logic, $x_2 = x_1(1 - \frac{1}{2021})$, $x_3 = x_2(1 - \frac{1}{2021})$, and so on (a geometric relation). Thus $x_i = x_0(1 - \frac{1}{2021})^i$, so $x_{2021} = x_0(1 - \frac{1}{2021})^{2021}$. However, we see that this is nearly equivalent to $x_0 \cdot \lim_{x\to\infty}(1 - \frac{1}{x})^x = x_0(\frac{1}{e})$. Since 2021 is such a large exponent, and we are only looking for the nearest integer, this is a reasonable approximation, leading to $x_{2021} \approx 136/e \approx 136/(2.72) = \boxed{50}$. Using a calculator we can confirm that the actual answer is about 50.02, so the approximation was reasonable.

14. The first antiderivative of $f(x) = x$ is $(1/2)x^2 + C$, the second antiderivative is $(1/6)x^3 + Cx + D$, and so on - in general, the nth antiderivative is $\frac{1}{(n+1)!}x^{n+1} + P(x)$, where P is some polynomial of degree $n - 1$. Thus the 2021st antiderivative is $g(x) = \frac{1}{2022!}x^{2022} + Cx^{2020} + Dx^{2019} + \ldots$. Letting the roots be r_1, r_2, and so on, we know that $(r_1 + r_2 + \cdots + r_{2022})(r_1 + r_2 + \cdots + r_{2022}) = (r_1^2 + r_2^2 + \cdots + r_{2022}^2) + 2(r_1r_2 + r_1r_3 + \cdots + r_{2021}r_{2022})$, so the sum of the squares of the roots is equal to the sum of the roots squared, minus twice the sum of the roots taken two at a time. Using Vieta's rules, we can see that the sum of the roots is 0 because the x^{2021} term has a coefficient of 0, but the sum two-at-a-time is $C(2022!)$. To find C, we take the 2020-th derivative to get $\frac{1}{2}x^2 + 2020!(C)$. If 1 is a root of this polynomial, then $C = -\frac{1}{2(2020!)}$. Multiplying this by 2022!, we see that the sum of the roots two-at-a-time is

$-1011(2021)$, so the sum of the squares is $0-2(-1011(2021)) = 2022(2021) = 2021^2+2021 = 4084441+2021 = \boxed{4086462}$.

15. The trick to this limit is swapping the $1/x$ variable with the x, which we can see is necessary because almost all the variables are $1/x$. If we let $h = 1/x$, then the new limit is $\lim_{h\to 0} \frac{1}{h}((h^2+4h+4)(2+h)^h - 4)$. Recognizing that $h^2+4h+4 = (h+2)^2$, we get $\lim_{h\to 0} \frac{1}{h}((2+h)^{2+h} - 4)$.
This seems remarkably close to the limit definition of the derivative, and if we replace 4 with 2^2 we see that this is just the derivative of $f(x) = x^x$ evaluated at $x = 2$. To find the derivative of $y = x^x$, we take the natural log of both sides for $\ln(y) = x\ln(x)$, then take the derivative of both sides for $\frac{y'}{y} = \ln(x)+1$. So $y' = y(\ln(x)+1) = x^x(\ln(x)+1)$. Evaluating this at $x = 2$ gives $\boxed{4\ln(2)+4}$. If you want, you can even test it by plugging a large number like 1000 into the original function - the limit does indeed converge to $4+4\ln 2$!

16. An infinitesimal percent change in Q can be written mathematically as $100\frac{dQ}{Q}$ where dQ is an infinitesimal absolute change in Q. Then, by the definition of price elasticity, $E = \frac{dQ/Q}{dP/P} = (P/Q)\frac{dQ}{dP}$. In the problem, $\frac{dQ}{dP} = -1/4$, the reciprocal of the derivative at 1, and $Q = 1$ while $P = 5-2(1^2) = 3$, meaning $E = \boxed{-0.75}$. This value can also be found by approximating with a small value of dQ.

17. If $(P/Q)\frac{dQ}{dP} = -2$ for all Q, then we can rearrange to get the differential equation $\frac{dQ}{dP} = -2Q/P$. Then, using separation of variables we get $\frac{1}{Q}dQ = -\frac{2}{P}dP$, and we can then integrate on both sides to get $\ln Q = -2\ln P + C \to Q = \frac{A}{P^2}$ for some value of A. We don't actually have to find A - all we need to know is that $Q_0 = A/(P_0)^2$, and that $P_1 = \frac{2}{3}P_0$, so $Q_1 = \frac{9}{4}Q_0 = \boxed{45}$.

18. Let the amplitude of Eric's sinusoidal motion, 21, be labelled A. The distance between Eric and Nitish is simply $D(x) = \sqrt{x^2 + A^2\sin^2(x))}$. If $D(x)$ is always increasing as Eric's x increases, we just need to find the value of x above which $D'(x)$ is always positive. $D'(x) = \dfrac{2x + 2A^2\sin(x)\cos(x)}{2D(x)}$. The denominator is always positive, so we are looking for the x value where the numerator switches from positive to negative. This requires solving $2x + A^2\sin(2x) = 0$ or $A^2\sin(2x) = -2x$. This is impossible to solve analytically, but instead we can consider the two graphs. Letting $2x = u$ for now, the graph of $A^2\sin(u)$ is a sine wave of amplitude A^2, while $-u$ is a line with slope -1. We don't know exactly where they intersect, but we do know that once $u > A^2$, they will never intersect again, because u has left the region in which $A^2\sin(u)$ is bounded. Additionally, tracing the two functions we see that as long as $u < A^2$, the two intersect at least once every period, so the actual final intersection could be anywhere between $A^2 - 2\pi$ and A^2. But this is the range for u - for $x = u/2$, the value can fall anywhere between $A^2/2 - \pi$ and $A^2/2$. Plugging in $A = 21$, we see that this is a range from $220.5 - \pi$ to 220.5. As a result, even though we do not know the exact value of x_0, to the nearest multiple of 10 it must be $\boxed{220}$. Graphing on Desmos shows that this analysis is correct, as $441\sin(2x) - 2x$ last crosses the x-axis at about $x = 217.6$, just within the π margin.

19. Let the equilateral triangle have vertices A, B, and C, where Vishnav is currently at A and D is the midpoint of BC. By symmetry, we can completely ignore AC and just focus on A, B, and D. Vishnav's motion can be decomposed into two different parts: running along side AB, and swimming to point D (due to the stipulation in

the problem, he cannot run along AB, swim to a point on BD, and then run to D). Lets say Vishnav stops running and starts swimming at a point E along AB. Let the distance EB be x. Then we can calculate the distance Vishnav runs as $4-x$, and the distance he swims, by Law of Cosines, as $\sqrt{x^2+2^2-2(2)(x)(1/2)}=\sqrt{x^2-2x+4}$. The total time that Vishnav takes to get to the car is then $T(x)=(4-x)/\sqrt{2}+(\sqrt{x^2-2x+4})/1$. We can find the minimum through $T'(x)=-\sqrt{2}/2+(x-1)/\sqrt{x^2-2x+4}=0 \rightarrow 2x-2=\sqrt{2x^2-4x+8} \rightarrow 4x^2-8x+4=2x^2-4x+8 \rightarrow 2x^2-4x-4=0 \rightarrow x^2-2x-2=0$. Using the quadratic formula, we find two roots at $1\pm\sqrt{3}$. Since the derivative is an upward-sloping parabola, it switches from negative to postiive at the greater root, indicating a relative minimum. Thus $T(x)$ is minimized at $x=1+\sqrt{3}$, so the minimum amount of time is $T(1+\sqrt{3})=(3-\sqrt{3})/\sqrt{2}+\sqrt{(1+\sqrt{3})^2-2(1+\sqrt{3})+4}=(3\sqrt{2}-\sqrt{6})/2+\sqrt{6}=\boxed{\dfrac{3\sqrt{2}+\sqrt{6}}{2}}$.

20. If $\arctan(1/x)>x/(1+x^2)$ for all positive x, and they are both always greater than 0, then the total area bound by them is just $\int_0^\infty \arctan(1/x)-x/(1+x^2)dx$. We can simplify $y=\arctan(1/x)$ by considering $\tan(y)=1/x$, so $x=\cot(y)=\tan(\pi/2-y)$. Therefore $\arctan(x)=\pi/2-y$, so $y=\pi/2-\arctan(x)$. Then the integral becomes $\int_0^\infty \pi/2-(\arctan(x)+x/(1+x^2))dx$. By inspection, $\arctan(x)+x/(1+x^2)$ is the product-rule derivative of $x\arctan(x)$, so the entire integral is just $\frac{\pi}{2}x-x\arctan(x)=x(\frac{\pi}{2}-\arctan(x))\Big|_0^\infty$. The lower bound clearly evaluates to 0, but the upper bound is indeterminate because x goes to infinity while $\pi/2-\arctan(x)$ goes to 0. To solve it, we can express it as $\dfrac{\pi/2-\arctan(x)}{1/x}$, which is still indeterminate (numerator and denominator both 0), then use L'hopital's rule. The derivative of the numerator is $-1/(1+x^2)$, while the derivative of the denominator is $-1/x^2$, leading to a new fraction of $\frac{x^2}{1+x^2}$. Ax x approaches infinity, this fraction clearly approaches $\boxed{1}$.

21. The series can be converted into $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}\cdots=\frac{3-1}{3!}+\frac{4-1}{4!}+\frac{5-1}{5!}\cdots=(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots)-(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots)$. Then, using the well-known fact that $e=1+1+1/2+1/(3!)+1/(4!)+\dots$, this becomes $S=(e-2)-(e-5/2)=\boxed{\dfrac{1}{2}}$.

22. Since the vertex is at the cake's right angle, we can call that the origin and then the equation of the parabolic curve is just $y=nx^2$ for some value of n. The two parabolas intersect at $nx^2=4-x^2 \rightarrow x=2/\sqrt{n+1}$. The total area of the cake is $\int_0^2(4-x^2)dx=4(2-0)-\frac{1}{3}(2^3-0^3)=\frac{16}{3}$. Then we must have $\int_0^{2/\sqrt{n+1}}4-(n+1)x^2dx=8/\sqrt{n+1}-\frac{n+1}{3}\left(\frac{8}{(n+1)^{3/2}}\right)=\frac{16}{3\sqrt{n+1}}=\frac{8}{3}$, so $n=3$ and the two parabolas intersect at $(1,3)$.

Then, to find the length of the cut we can use arc-length formula: $L=\int_0^1\sqrt{1+(6x)^2}dx$. Letting $u=6x$, $L=\frac{1}{6}\int_0^6\sqrt{1+u^2}du$. Next, we can use the trig substitution $u=\tan\theta$, transforming the integral from $\int_0^6\sqrt{1+u^2}du=\int_0^{\arctan 6}\sec^3\theta d\theta$. The $\sec^3\theta$ integral can be evaluated easily with integration by parts with $u=\sec\theta$ and $dv=\sec^2\theta$: $I=\sec\theta(\tan\theta)-\int\sec\theta\tan\theta(\tan\theta)d\theta$. That second integral becomes $\int\sec\theta\tan^2\theta d\theta=\int\sec^3\theta-\sec\theta d\theta=I+\int\sec\theta d\theta$. Shifting the I to the left side, we get $2I=\sec\theta\tan\theta+\int\sec\theta d\theta$. The integral of $\sec\theta$ is quite common and can be derived by multiplying by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$ as $\ln(\sec\theta+\tan\theta)$. Therefore $I=\frac{\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)}{2}$. Finally, we can use a right triangle to see that the angle with tangent of 6 has a secant of $\sqrt{37}$, so when $\arctan(6)$ is plugged in, and everything is divided by 6 then multiplied by 12, the final answer is $\boxed{6\sqrt{37}+\ln(6+\sqrt{37})}$

23. Evaluating all these volumes is tedious with disk method, but Pappus's theorem, which says that $V=2\pi Ar$ where r is the distance from the center of mass to the line of rotation, makes it much easier. All we have to

do is find the COM and then rank the lines by their distance from the COM. The x-coordinate of the COM is $\frac{1}{A}\int_0^2 x(4-x^2)dx = \frac{1}{A}(2x^2 - \frac{1}{4}x^4)\Big|_0^2 = 4/A = \frac{3}{4}$ where we used the fact that the cake's area is 16/3 which was found in the previous solution. Similarly, the y-coordinate of the COM is $\frac{1}{A}\int_0^4 y\sqrt{4-y}dy$. To solve this integral, we set $u = 4-y$ to get $\frac{1}{A}\int_0^4 (4-u)\sqrt{u}du = \frac{1}{A}(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2})\Big|_0^4 = \frac{8}{5}$.

Now we just find the distance from the COM, $(3/4, 8/5)$, to each of the lines. The distance from $x = 0$ is $3/4$, the distance from $y = 0$ is $8/5$, the distance from $x = 4$ is $13/4$, and the distance from $y = 5-x$ can be computed from point-to-line formula: standard form is $x + y - 5 = 0$, so we have $d = |(3/4 + 8/5 - 5)|/\sqrt{2} = 53/(20\sqrt{2})$. We know that $26.5 < 20\sqrt{2} < 30$ because $1.325 < \sqrt{2} < 1.5$, so $53/30 < 53/(20/\sqrt{2}) < 53/26.5$ and $1.6 < 53/(20/\sqrt{2}) < 2$. Combining this with the other values, it's clear the ordering is $\boxed{D < P < F < M}$.

24. Using the area found in #22 and the COM found in #23, we can easily calculate the volume of the whole cake as $2\pi(16/3)(3/4) = 8\pi$. Now we are looking for the rectangular prism of maximum volume that can be inscribed in this figure, with one face on the circular base. To do this, we can write an explicit equation for the shape, which is easiest to do in cylindrical coordinates r (distance from origin along the circular base), θ (angle from horizontal along the circular base), and z (height above circular base). Consider the point (r, θ). If we cut the cake along the plane corresponding to θ, we form a cross-section that is just the parabola $z = 4 - r^2$. Then at every value of r, $z = 4 - r^2$, independent of θ. So if one vertex of the rectangular base is at (r, θ), the volume of the quarter-prism it forms is $xyz = (r\cos\theta)(r\sin\theta)(4-r^2) = \frac{1}{2}\sin(2\theta)(4r^2 - r^4)$. Treating θ and r independently, V is maximized when $\sin(2\theta) = 1$, which only occurs when $\theta = \pi/4$; V is also maximized when $4r^2 - r^4$ is maximized, so $8r - 4r^3 = 0 \to r(2-r^2) = 0$. The derivative switches from positive to negative at $r = \sqrt{2}$, so that is the maximizing value of r. Then we can find the total volume of the bite as $4xyz = 2\sin(2\theta)(4r^2 - r^4) = 2(1)(4) = 8$. The ratio of this volume to the total volume of the cake is $8/8\pi = \boxed{\frac{1}{\pi}}$.

25. Let the bug's direction be characterized by a vector with angle θ from the horizontal. By symmetry, the probability that the bug leaves the square when $0 < \theta < \pi/2$ is equal to the probability that it leaves the square for any θ. Let X be the event that the bug *does not* escape from the square. $P(X) = \sum_{(0,\pi/2)} P(X|\theta) * P(\theta)$ by conditioning over the angle. For a certain angle θ in $(0, \pi/2)$, the probability that the bug travels at that angle is $\frac{d\theta}{\pi/2}$. In other words, since the angle is continuous, we can replace the summation with an integral: $P(X) = \frac{2}{\pi}\int_0^{\pi/2} P(X|\theta)d\theta$. Given an angle θ, we must now find the total area of starting locations for which the bug does not leave the square after traveling 1 m in that direction. The horizontal motion of the bug will be $\cos\theta$ and the vertical motion will be $\sin\theta$, so the bug can only stay within the square if it starts within a smaller rectangle of width $1 - \cos\theta$ and height $1 - \sin\theta$ (you can see this by diagramming it). Then the probability $P(X|\theta)$ is the ratio of the area of this rectangle to the area of the unit square, or $(1-\cos\theta)(1-\sin\theta) = \sin\theta\cos\theta - \sin\theta - \cos\theta + 1$. Plugging into the integral, we get $\int_0^{\pi/2} \frac{1}{2}\sin(2\theta) - \sin\theta - \cos\theta + 1 d\theta = \frac{\pi-3}{2}$. Then we multiply back in the $2/\pi$ to get $1 - 3/\pi$. However, this is $P(X)$, and we are looking for $P(X^C) = 1 - P(X) = \boxed{\frac{3}{\pi}}$

26. The given statement by Relpek implies that $A(a, a+\tau)$ is a function of τ. However, note that polar integration tells

us that this area is

$$\int_a^{a+\tau} \frac{1}{2} r(t)^2 d(\theta(t)) = \int_a^{a+\tau} \frac{1}{2}(\theta(t))^4 \cdot \theta'(t)dt = \frac{1}{10}(\theta(t))^5 \Big|_a^{a+\tau} = \frac{1}{10}\left((\theta(a+\tau))^5 - (\theta(a))^5\right)$$

By setting $a = 0$, we get that $\frac{1}{10}\left((\theta(a+\tau))^5 - (\theta(a))^5\right) = \frac{1}{10}\left((\theta(\tau))^5 - (\theta(0))^5\right)$ for all a, τ. Define the function $g(r) = (\theta(r))^5 - (\theta(0))^5$. The equation above now boils down to $g(a+\tau) = g(a) + g(\tau)$. This means g is additive, and since θ is differentiable, g is differentiable too. Cauchy showed that any additive continuous function must be linear, meaning that $g(a) = Ca$ for some constant C (since $g(0) = g(0) + g(0) \implies g(0) = 0$). This means that $\theta(t) = \sqrt[5]{Ct + D}$ for constants C, D. Now, it is a matter of using the remaining conditions to find C, D. Since $\theta(0) = \pi$, then $D = \pi^5$. Now, note that

$$-2\pi^2 = r'(0) = 2\theta(0)\theta'(0) = 2\pi \cdot \frac{1}{5}(C \cdot 0 + \pi^5)^{-4/5} \cdot C = \frac{2\pi}{5\pi^4} \cdot C \implies C = -5\pi^5$$

This means that we want to find the smallest positive solution to $\theta(t) = \sqrt[5]{-5\pi^5 t + \pi^5} = \pi\sqrt[5]{1 - 5t} = \frac{\pi}{2}$, which means $1 - 5t = \frac{1}{32} \implies t = \boxed{\frac{31}{160}}$

27. First let $u = x - a$. This gives $(\ln 1000)^{1001} \int_0^\infty \frac{u^{100}}{100^{u+a}} du = \frac{(\ln 1000)^{1001}}{1000^a} \int_0^\infty \frac{u^{1000}}{e^{u \ln 1000}} du$. Then letting $v = u \ln 1000$, we have $\frac{(\ln 1000)^{1000}}{1000^a} \int_0^\infty (v/\ln 1000)^{1000} e^{-v} dv = 1000^{-a} \int_0^\infty v^{1000} e^{-v} dv$. This integral can be solved through tabular integration with differentiating term v^{1000} and integrating term e^{-v}. The series is $-v^{1000}e^{-v} - 1000v^{999}e^{-v} - (1000)(999)v^{998}e^{-v} - \cdots - (1000!)ve^{-v} + (1000!)\int_0^\infty e^{-v} dv$. Plugging in bounds we see that all but the final term go to 0 on both ends due to the v and e^{-v} factors. Thus the entire integral evaluates to $-(1000!)(e^{-\infty} - e^0) = 1000!$. Bringing back the outside factor we have $(1000!)/(1000^a)$. For this to be an integer, $1000^a = 10^{3a}$ must be a factor of 1000!. To find the number of factors of 10 in 1000!, we must find the number of factors of 5 (because there are far more factors of 2 than 5). To find the number of factors of 5 we add the number of multiples of 5 less than 100 to the number of multiples of 25 to the number of multiples of 125 to the number of multiples of 625: $1000/5 + 1000/25 + 1000/125 + 1 = 249$. The largest a such that $3a$ is less than or equal to 249 is simply $249/3 = \boxed{83}$.

28. First let the radius of the circle be r and the length of the leash be l. Using the hint, we consider the wrapping of Tanmay's leash according to the point at which it contacts the circular pen. Consider only one half of the circle as well (we will double at the end). First Tanmay can graze a quarter circle of radius l before the leash begins to collide with the circle. Then, the contact point will begin to move along the quarter circle. Consider the angle θ between the tying point, the circle's center, and the contact point. If the contact point is at an angle θ then $r\theta$ meters of leash have already been "used up" in wrapping around the circle. While at that contact point, the remaining length of the leash $l - r\theta$ rotates through a small angle $d\theta$. The leash sweeps through a circular arc of radius $l - r\theta$ and angle $d\theta$, with an area of $\frac{1}{2}(l - r\theta)^2 d\theta$. But, since this occurs at all angles θ, all we have to do is integrate it from $\theta = 0$ to $\theta = \pi$! (the leash extends exactly to $\theta = \pi$ because it is exactly half the circumference in length). So we integrate $\frac{1}{2}\int_0^\pi (l - r\theta)^2 d\theta = \frac{1}{2}(l^2\theta - lr\theta^2 + \frac{1}{3}r^2\theta^3) = \frac{1}{2}(l^2\pi - lr\pi^2 + \frac{1}{3}r^2\pi^3) = \frac{1}{2}(\pi^3 - \pi^3 + \frac{1}{3}\pi^3) = \frac{1}{6}\pi^3$. Then we add in the area of the quarter circle to get $\frac{1}{4}\pi^3 + \frac{1}{6}\pi^3 = \frac{5}{12}\pi^3$. Lastly, we double this to account for both sides of the pen, giving $\boxed{\frac{5}{6}\pi^3}$.

29. First we use the substitution $u = e^x$, which gives $\int_1^2 e^u/u du$. This integral is impossible to solve analytically, but we can try to bound it using Riemann sums. First off, we can take the derivative of $e^u/u = \frac{ue^u - e^u}{u^2} = (e^u/u^2)(u-1)$, which shows that the function is increasing over the entire interval (whenever $u > 1$). Since the function is increasing, a right Riemann Sum will be an overestimate and a left Riemann sum will be an underestimate. However, we cannot use too many subintervals because it is difficult to work with non-integer powers of e - instead we will only use one rectangle for each. The right sum with one interval is just $wh = 1(e^2/2) \approx (2.7)^2/2 = 7.29/2 = 3.645$. The left sum with one interval is $wh = 1(e) \approx 2.72$. Thus we can be sure that the integral is between 2.72 and 3.65, narrowing the options down to either 3 or 4 as the closest integer. However, we can get better! Using a trapezoidal sum with one interval, we get $h(b_1 + b_2)/2 = 1(e + e^2/2)/2 \approx 1(2.72 + 3.65)/2 = (6.37)/2 = 3.19$. This narrows the range down to 2.72 to 3.19, so we can be sure the closest integer is $\boxed{3}$! This problem can also be solved by expanding into the Taylor series and taking the first terms, but it is a bit more tedious.

30. Looking back to the first question, we see that the Weierstrass substitution is $u = \tan(x/2)$. Back then we derived the relations $\sin(x) = \frac{2u}{1+u^2}$ and $\frac{2}{1+u^2}du = dx$. Plugging these in, the integral simplifies to $\int_0^1 \frac{4u}{(1+u)^2(1+u^2)}du$. We can expand the integrand into partial fraction decomposition: $\frac{4u}{(1+u)^2(1+u^2)} = \frac{Au+B}{u^2+2u+1} + \frac{Cu+D}{1+u^2}$. Cross-multiplying, we can form 4 separate equations for each power of u.

u^3: $C + A = 0$

u^2: $B + 2C + D = 0$

u: $A + C + 2D = 4$

1: $B + D = 0$

Recognizing from equations 1 and 4 that $A = -C$ and $B = -D$, we can show that $A = C = 0$ from equation 2, and thus that $D = 2$ and $B = -2$. Therefore the integral simplifies to $\int_0^1 \frac{2}{1+u^2} - \frac{2}{(1+u)^2}du$. Using standard integration techniques, this becomes $2\arctan(u) + 2/(1+u) \rightarrow \boxed{\frac{\pi}{2} - 1}$.