1. **Answer: C** $\left(80\sqrt{3}+48\right)$
Let x be the width of the inner rectangle and y be the length of the inner rectangle. This means that $x+6$ is the width of the outer rectangle and $y+8$ is the length of the outer rectangle. The area of the inner rectangle is xy, and the area of the outer rectangle is $(x+6)(y+8) = xy+6y+8x+48$. Let A be the area of the region between the two rectangles, which is equal to $(xy+6y+8x+48)-(xy) = 6y+8x+48$. We are given that $xy = 100$, which means that $x = \frac{100}{y}$. We can substitute this into our expression for A to solve for the value of y, which must be positive, that minimizes A. This gives us $A = 6y + \frac{800}{y} + 48$. We can then determine that $\frac{dA}{dy} = 6 - \frac{800}{y^2} = 0$ when $y = \sqrt{\frac{400}{3}} = \frac{20\sqrt{3}}{3}$. Thus, $x = \frac{100}{\left(\frac{20\sqrt{3}}{3}\right)} = 5\sqrt{3}$. Therefore, $A = 6\left(\frac{20\sqrt{3}}{3}\right) + 8(5\sqrt{3}) + 48 = \boxed{80\sqrt{3}+48}$ square inches.

2. **Answer: A** (3)
We can determine that $y' = \frac{2}{x}$, $y'' = \frac{-2}{x^2}$, and $y''' = \frac{4}{x^3}$. We can then substitute these values into the equation to get $\frac{-6}{x^2} - \frac{4}{x^3} + \frac{2}{x} = \frac{2x^2-6x-4}{x^3} = 0$. Our solutions for x are when $2x^2-6x-4=0$, and since both of the solutions to that equation are real, the sum of the solutions by Vieta's is $\frac{-(-6)}{2} = \boxed{3}$.

3. **Answer: B** $\left(\sqrt{2}-1\right)$
We can first simplify this limit as the following:

$$\lim_{y\to\infty}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sqrt[y]{\frac{1+\tan^y(x)}{\sec^y(x)}}\right)dx = \cos(x)\lim_{y\to\infty}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sqrt[y]{1+\tan^y(x)}\right)dx.$$

We can divide the bounds of this integral into 2 smaller intervals. Consider the value of $\tan(x)$ on the intervals $\frac{\pi}{6} \le x < \frac{\pi}{4}$ and $\frac{\pi}{4} \le x \le \frac{\pi}{3}$. For the first interval, $\frac{\sqrt{3}}{3} \le \tan(x) < 1$, and for the second interval, $1 \le \tan(x) \le \sqrt{3}$. This means that for the first interval

$$\lim_{y\to\infty}\left(\sqrt[y]{1+\tan^y(x)}\right) = 1,$$

and for the second interval

$$\lim_{y\to\infty}\left(\sqrt[y]{1+\tan^y(x)}\right) = \tan(x).$$

Therefore,

$$\cos(x)\lim_{y\to\infty}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sqrt[y]{1+\tan^y(x)}\right)dx = \cos(x)\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(1)\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\tan(x))\ dx\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cos(x))\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sin(x))\ dx =$$

$$\sin(x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \cos(x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\sqrt{2}-1}{2} + \frac{\sqrt{2}-1}{2} = \boxed{\sqrt{2}-1}.$$

4. **Answer: D** $\left(\frac{-1}{2204496}\right)$
The Maclaurin representation for $f(x)$ is

$$f(x) = 5x^{10}\sum_{n=0}^{\infty}\frac{\left(-\frac{x^4}{3}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{5x^{10+4n}}{n!(-3)^n}.$$

The x^{38} term occurs when $n=7$, and so the coefficient of this term is $\frac{5}{7!(-3)^7} = \boxed{\frac{-1}{2204496}}$.

5. **Answer: C** $\left(\frac{-187}{2}\right)$
By the Chain Rule, $h'(x) = \left(\frac{-1}{2}\right)g'\left(f\left(\frac{-x}{2}\right)\right)f'\left(\frac{-x}{2}\right) = \left(\frac{-1}{2}\right)g'\left(\frac{5x^2+3}{2}\right)f'\left(\frac{-x}{2}\right)$. Thus, $h'(-4) = \left(\frac{-1}{2}\right)g'\left(\frac{83}{2}\right)f'(2) = \left(\frac{-1}{2}\right)(5)(37) = \boxed{\frac{-185}{2}}$.

6. **Answer: D** $(65 - 32\pi)$
We can first find the values of θ where the curves intersect. If we let $8 = 8 - 2\cos\theta$, then $\cos\theta = 0$. This is true when $\theta = \frac{-\pi}{2}, \frac{\pi}{2}$. Thus, our desired area is equal to

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{(8-2\cos\theta)^2}{2}\right) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{8^2}{2}\right) d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(32 + 2\cos^2\theta - 16\cos\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (32)\ d\theta =$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (33 + \cos 2\theta - 16\cos\theta)\ d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (32)\ d\theta = \left(33\theta + \frac{\sin 2\theta}{2} - 16\sin\theta\right)\Bigg|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + (32\theta)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{65\pi - 32}.$$

Note: If the region of intersection of the curves is not intuitive to you at first, then it may be helpful for you to draw a quick sketch.

7. **Answer: A** $\left(\frac{3\sqrt{3}}{4}\right)$
We first need to identify the region bounded by the two curves. Setting $y^2 = \frac{7-x}{4} = \frac{x-4}{2}$ indicates to us that the two curves intersect at two points where $x = 5$. A quick sketch indicates to us that our desired volume is the following:

$$\frac{\sqrt{3}}{4}\left(\int_4^5 \left(2\sqrt{\frac{x-4}{2}}\right)^2 dx + \int_5^7 \left(2\sqrt{\frac{7-x}{4}}\right)^2 dx\right) = \frac{\sqrt{3}}{4}\left(\int_4^5 (2(x-4))\ dx + \int_5^7 (7-x)\ dx\right) =$$

$$\frac{\sqrt{3}}{4}\left(\left(x^2 - 8x\right)\Bigg|_4^5 + \left(\frac{-x^2}{2} + 7x\right)\Bigg|_5^7\right) = \frac{\sqrt{3}}{4}(1+2) = \boxed{\frac{3\sqrt{3}}{4}}.$$

8. **Answer: C** $\left(\ln\left(\frac{1}{4}\right)\right)$
We want to check the value of $f(x)$ at any critical points and endpoints (even though -2 and -1 are not included in the domain). $f'(x) = \frac{-2x-3}{-x^2-3x-2} = 0$ when $x = \frac{-3}{2}$ and $f(x) = \ln\left(\frac{1}{4}\right)$. $f''\left(\frac{-3}{2}\right) = \frac{-2(\frac{-3}{2})^2 - 6(\frac{-3}{2}) - 5}{(-(\frac{-3}{2})^2 - 3(\frac{-3}{2}) - 2)^2} < 0$. This means that $\left(\frac{-3}{2}, \ln\left(\frac{1}{4}\right)\right)$ is the only critical point for $f(x)$ on our domain, and since $f(x)$ is concave down at this point and along the entire domain, $\left(\frac{-3}{2}, \ln\left(\frac{1}{4}\right)\right)$ is a maximum and we do not need to check any potential "endpoints." Thus, our answer is $\boxed{\ln\left(\frac{1}{4}\right)}$.

9. **Answer: E** (729)
There are many ways to simplify this function before evaluating the derivative. Perhaps the quickest method is to note that $f(x) = \cos 3x - \sin 3x$. From here we can determine that $f^{(6)}(x) = 729(\sin 3x - \cos 3x)$ and $f^{(6)}\left(\frac{\pi}{3}\right) = 729(0+1) = \boxed{729}$.

10. **Answer: E** (1)
Let μ represent the mean value of this distribution.

$$\mu = \int_{-\infty}^{\infty} (xf(x))\ dx = \int_0^{\infty} \left(xe^{-x}\right) dx.$$

To solve this integral we can use Integration By Parts. If we let $u = x$ and $dv = e^{-x}dx$, then $du = dx$ and $v = -e^{-x}$.

$$\int_0^{\infty} \left(xe^{-x}\right) dx = -xe^{-x}\Bigg|_0^{\infty} - \int_0^{\infty} \left(-e^{-x}\right) dx = \boxed{1}.$$

11. **Answer: B** $\left(\frac{11}{2}\right)$
The fifth real root of $f(x)$ will be the value of x_5 in the following equation where x_i represents a distinct root of $f(x)$:

$$\sum_{i=1}^{5} \frac{1}{5 - x_i} = \frac{1}{5-(-1)} + \frac{1}{5-(2)} + \frac{1}{5-(3)} + \frac{1}{5-(4)} + \frac{1}{5-(x_5)} = 2 + \frac{1}{5-x_5} = \frac{f'(5)}{f(5)} = 0.$$

After solving this equation, we get that $x_5 = \boxed{\frac{11}{2}}$.

12. **Answer: D** $\left(\frac{2673}{8960}\right)$
Note that

$$\frac{d}{dx}\left(\prod_{y=1}^{10}\left(\frac{yx+3}{y}\right)\right) = \frac{d}{dx}\left(\prod_{y=1}^{10}\left(x+\frac{3}{y}\right)\right).$$

It may be helpful to first write out some of the terms of this derivative. We can use the Product Rule to simplify this derivative.

$$\text{Let } P(x) = \prod_{y=1}^{10}\left(x+\frac{3}{y}\right) = (x+3)\left(x+\frac{3}{2}\right)...\left(x+\frac{3}{10}\right).\ \frac{d}{dx}\left(P\left(x\right)\right) = \sum_{y=1}^{10}\frac{P\left(x\right)}{\left(x+\frac{3}{y}\right)}.$$

$$\text{At } x=0,\ \frac{d}{dx}\left(P\left(x\right)\right) = \sum_{y=1}^{10}\frac{\frac{3^{10}}{10!}}{\left(\frac{3}{y}\right)} = \sum_{y=1}^{10}\frac{3^9 y}{10!} = \frac{3^9\left(55\right)}{10!} = \boxed{\frac{2673}{8960}}.$$

13. **Answer: E** $\left(e^3-e\right)$

$$\lim_{y\to\infty}\int_1^3\left(\frac{y+x+4}{y+4}\right)^{y+3}dx = \lim_{y\to\infty}\int_1^3\left(1+\frac{x}{y+4}\right)^{y+3}dx = \lim_{y\to\infty}\left(1+\frac{x}{y+4}\right)^{y+4}\Bigg|_1^3 =$$

$$\lim_{y\to\infty}\left(1+\frac{3}{y+4}\right)^{y+4} - \lim_{y\to\infty}\left(1+\frac{1}{y+4}\right)^{y+4} = \boxed{e^3-e}.$$

14. **Answer: C** $\left(\frac{\pi\sqrt{3}}{6}\right)$
Let $\cos\left(x\right) = \frac{1-u^2}{1+u^2}$ and $dx = \left(\frac{2}{1+u^2}\right)du$ (this comes from the Weierstrass substitution).

$$\int_0^{\frac{\pi}{2}}\left(\frac{3}{4+2\cos\left(x\right)}\right)dx = 3\int_0^1\left(\frac{1}{u^2+3}\right)du = 3\left(\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{u\sqrt{3}}{3}\right)\Bigg|_0^1\right) = \boxed{\frac{\pi\sqrt{3}}{6}}.$$

15. **Answer: C** $\left(\frac{25}{108}\right)$

$$\sum_{n=0}^{\infty}\frac{n^2+7n}{7^{n+1}} = \sum_{n=1}^{\infty}\frac{(n-1)^2+7(n-1)}{7^{(n-1)+1}} = \sum_{n=1}^{\infty}\frac{n^2+5n-6}{7^n} = \sum_{n=1}^{\infty}\frac{n^2}{7^n} + \sum_{n=1}^{\infty}\frac{5n}{7^n} + \sum_{n=1}^{\infty}\frac{-6}{7^n}.$$

$$\text{Let } S_1 = \sum_{n=1}^{\infty}\frac{n^2}{7^n}, S_2 = \sum_{n=1}^{\infty}\frac{5n}{7^n}, \text{and } S_3 = \sum_{n=1}^{\infty}\frac{-6}{7^n}.$$

$$\frac{S_1}{7} = \sum_{n=1}^{\infty}\frac{n^2}{7^{n+1}} = \sum_{n=0}^{\infty}\frac{n^2}{7^{n+1}}.$$

$$\frac{S_2}{35} = \sum_{n=1}^{\infty}\frac{n}{7^{n+1}} = \sum_{n=1}^{\infty}\frac{n+1}{7^{n+1}} - \sum_{n=1}^{\infty}\frac{1}{7^{n+1}} = \sum_{n=1}^{\infty}\frac{n}{7^n} - \frac{1}{7} - \frac{\frac{1}{49}}{1-\frac{1}{7}} = \frac{S_2}{5} - \frac{1}{6}, \text{which means that } S_2 = \frac{35}{36}.$$

$$S_3 = \sum_{n=1}^{\infty}\frac{-6}{7^n} = \frac{\frac{-6}{7}}{1-\frac{1}{7}} = -1.$$

$$\sum_{n=0}^{\infty}\frac{n^2+7n}{7^{n+1}} = \frac{S_1}{7} + \frac{S_2}{5} = \frac{S_1}{7} + \frac{7}{36} = S_1 + \frac{35}{36} - 1, \text{which means that } S_1 = \frac{7}{27}.\ \text{Thus, } \sum_{n=0}^{\infty}\frac{n^2+7n}{7^{n+1}} = \boxed{\frac{25}{108}}.$$

16. **Answer: B** $\left(135\pi^2\right)$

Our volume is equal to the value of the definite integral $\pi\displaystyle\int_0^{2\pi} y^2\,dx$.

$$\pi\int_0^{2\pi} y^2\,dx = \pi\int_0^{2\pi}(3-3\cos(t))^2(3-3\cos(t))\,dt = 27\pi\int_0^{2\pi}(1-\cos(t))^3\,dt =$$

$$27\pi\int_0^{2\pi}\left(1-3\cos(t)+3\cos^2(t)-\cos^3(t)\right)dt = 27\pi\int_0^{2\pi}\left(\frac{5}{2}-\frac{15\cos(t)}{4}+\frac{3\cos(2t)}{2}-\frac{\cos(3t)}{4}\right)dt =$$

$$27\pi\left(\frac{5t}{2}-\frac{15\sin(t)}{4}+\frac{3\sin(2t)}{4}-\frac{\sin(3t)}{12}\right)\Bigg|_0^{2\pi} = 27\pi(5\pi) = \boxed{135\pi^2}.$$

17. **Answer: A** $\left(\left(\frac{11}{8},\frac{-21}{5}\right)\right)$
Since $y = x^2-9 = (x-3)(x+3)$, our curve intersects the positive x-axis at $(3,0)$, which is A, and it intersects the y-axis at $(0,-9)$, which is B. In order to find C, we should first compute the area of the region. We can treat our region as the region bounded by $y=0$ and $y=x^2-9$ from $x=0$ to $x=3$. The area of this is equal to

$$\int_0^3\left((0)-\left(x^2-9\right)\right)dx = \left(\frac{-x^3}{3}+9x\right)\Bigg|_0^3 = 18.$$

We now need to find $\overline{x}$ and $\overline{y}$.

$$\overline{x} = \frac{1}{(18)}\int_0^3 x\left((0)-\left(x^2-9\right)\right)dx = \frac{1}{18}\int_0^3\left(-x^3+9x\right)dx = \frac{1}{18}\left(\frac{-x^4}{4}+\frac{9x^2}{2}\right)\Bigg|_0^3 = \frac{9}{8}.$$

$$\overline{y} = \frac{1}{(18)}\int_0^3\frac{1}{2}\left((0)^2-\left(x^2-9\right)^2\right)dx = \frac{1}{36}\int_0^3\left(-x^4+18x^2-81\right)dx = \frac{1}{36}\left(\frac{-x^5}{5}+6x^3-81x\right)\Bigg|_0^3 = \frac{-18}{5}.$$

Therefore, C has coordinates of $\left(\frac{9}{8},\frac{-18}{5}\right)$. The centroid of ΔABC can be computed using the above method, but for a triangle, a quicker way is to find the averages of the x-coordinates and the y-coordinates of the vertices of the triangle. Thus, the centroid of ΔABC lies at $\left(\frac{3+0+\frac{9}{8}}{3},\frac{0+(-9)+\left(\frac{-18}{5}\right)}{3}\right) = \boxed{\left(\frac{11}{8},\frac{-21}{5}\right)}$.

18. **Answer: B** (40)
For any x, the area of this triangle, A, is equal to $\frac{(5x-3x)(x)}{2} = x^2$. Let t represent time in seconds. $\frac{dA}{dt} = 2x\frac{dx}{dt}$. Thus, when $x=10$, $\frac{dA}{dt} = 2(10)(2) = \boxed{40}$ square units per second.

19. **Answer: A** (12)

We first need to compute the surface area of Rayyan's yacht, which can be represented as $\displaystyle\int 2\pi x\,ds$. Since $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = -4$, then $ds = \sqrt{(2)^2+(-4)^2}\,dt = 2\sqrt{5}\,dt$. Thus, the surface area is equal to $\displaystyle\int_1^3 2\pi(2t+4)\,2\sqrt{5}\,dt =$

$$4\pi\sqrt{5}\left(t^2+4t\right)\Bigg|_1^3 = 64\pi\sqrt{5}\text{ square units.}$$

To solve for x, we can write the equation $2\pi\sqrt{5}(x)+5\left(2\pi\sqrt{5}+6\pi\sqrt{5}\right) = 64\pi\sqrt{5}$. Solving for x, we get that $x = \frac{64\pi\sqrt{5}-5\left(2\pi\sqrt{5}+6\pi\sqrt{5}\right)}{2\pi\sqrt{5}} = \boxed{12}$ minutes.

20. **Answer: D** $\left(\frac{\pi}{3}\right)$

We should first note that $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot f(x_i)$ where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$.

$$\sum_{i=1}^{n} \frac{\sqrt{5}}{\sqrt{20n^2 - 5\left(i+n\right)^2}} = \sum_{i=1}^{n} \frac{\sqrt{5}}{2n\sqrt{5 - \left(\frac{\sqrt{5}}{2n}\left(i+n\right)\right)^2}} = \sum_{i=1}^{n} \frac{\frac{\sqrt{5}}{2n}}{\sqrt{5 - \left(\frac{\sqrt{5}}{2} + \frac{i\sqrt{5}}{2n}\right)^2}} = \sum_{i=1}^{n} \frac{\frac{\sqrt{5}-\frac{\sqrt{5}}{2}}{n}}{\sqrt{5 - \left(\frac{\sqrt{5}}{2} + \frac{i\sqrt{5}}{2n}\right)^2}}.$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{\sqrt{5}}{\sqrt{20n^2 - 5\left(i+n\right)^2}}\right) = \lim_{t\to\sqrt{5}^-}\int_{\frac{\sqrt{5}}{2}}^{t} \frac{1}{\sqrt{5-x^2}}\,dx = \lim_{t\to\sqrt{5}^-}\left(\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right)\Bigg|_{\frac{\sqrt{5}}{2}}^{t} = \boxed{\frac{\pi}{3}}.$$

21. **Answer: B** (I and III only)

Let's look at each series separately.

I : Let's first check for absolute convergence. $\displaystyle\sum_{n=1}^{\infty}\left|\frac{-2\cos\left(n\right)}{7n^2}\right| = \sum_{n=1}^{\infty}\left|\frac{2\cos\left(n\right)}{7n^2}\right|.$

Note that $0 \le \left|\cos\left(n\right)\right| \le 1$. We know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test.

Thus, because $0 \le \left|\dfrac{2\cos\left(n\right)}{7n^2}\right| \le \dfrac{2}{7n^2}$ for $n \ge 1$, then $\displaystyle\sum_{n=1}^{\infty}\left|\frac{2\cos\left(n\right)}{7n^2}\right|$ converges by the Direct Comparison Test and series I converges absolutely.

II : Let's first check for absolute convergence. $\displaystyle\sum_{n=10}^{\infty}\left|\frac{(-1)^{n+1}}{\ln\left(n\right)}\right| = \sum_{n=10}^{\infty}\frac{1}{\ln\left(n\right)}.$

Consider the series $\displaystyle\sum_{n=10}^{\infty}\frac{1}{n\ln\left(n\right)}$. This series diverges by the Integral Test because $\displaystyle\int_{10}^{\infty}\frac{1}{n\ln\left(n\right)}\,dn = \lim_{x\to\infty}\ln\left|\ln\left(n\right)\right|\Big|_{10}^{\infty} = +\infty$. Thus, because $0 < \dfrac{1}{n\ln\left(n\right)} < \dfrac{1}{\ln\left(n\right)}$ for $n \ge 10$, then $\displaystyle\sum_{n=10}^{\infty}\frac{1}{\ln\left(n\right)}$ diverges by the Direct Comparison Test and series II does not converge absolutely.

III : Let's first check for absolute convergence. $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n\left(3n^6+14\right)}{7n^{10}-5n^2}\right| = \sum_{n=2}^{\infty}\frac{3n^6+14}{7n^{10}-5n^2}.$

We know that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^4}$ converges by the p-series test. $\displaystyle\lim_{n\to\infty}\left[\frac{3n^6+14}{7n^{10}-5n^2}\cdot\frac{n^4}{1}\right] = \lim_{n\to\infty}\left[\frac{3n^{10}+14n^4}{7n^{10}-5n^2}\right] = \frac{3}{7}.$

Thus, by the Limit Comparison Test, series III converges absolutely.

Therefore our answer is $\boxed{I \text{ and } III \text{ only}}$.

22. **Answer: D** $\left(\frac{1}{4}\left(\frac{3\sqrt{2}-2}{3}+\ln\left(\frac{\sqrt{6}+\sqrt{3}}{3}\right)\right)\right)$

$$\text{This length is equal to } \int_{\frac{\sqrt{3}}{3}}^{1}\sqrt{\left(\frac{\theta}{2}\right)^2+\left(\frac{1}{2}\right)^2}\,d\theta=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1}\sqrt{\theta^2+1}\,d\theta.$$

Let $\theta=\tan x$.

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sqrt{\theta^2+1}\,d\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\sqrt{\tan^2 x+1}\right)\sec^2 x\,dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^3 x\,dx.$$

We can evaluate this integral using Integration By Parts. If we let $u=\sec x$ and $dv=\sec^2 x\,dx$, then $du=\sec x\tan x\,dx$ and $v=\tan x$.

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^3 x\,dx=\frac{1}{2}\left(\sec x\tan x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec x\tan^2 x\,dx\right)=\frac{1}{2}\left(\frac{3\sqrt{2}-2}{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\sec^3 x-\sec x\right)\,dx\right)=$$

$$\frac{1}{2}\left(\frac{3\sqrt{2}-2}{3}+\ln\Big|\sec x+\tan x\Big|\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}\right)-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\sec^3 x\right)\,dx.$$

After simplifying this equation, we can determine that

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^3 x\,dx=\boxed{\frac{1}{4}\left(\frac{3\sqrt{2}-2}{3}+\ln\left(\frac{\sqrt{6}+\sqrt{3}}{3}\right)\right)}.$$

23. **Answer: C** $\left(\frac{7803}{55}\right)$

Let's label the three lines as $L1, L2$, and $L3$ based on the order they were written in the problem.
L1: We first compute that $\frac{dy}{dx}[x^2+xy+y^2=27]$ is equal to $2x+y+xy'+2yy'=0$, which can be simplified in the form $y'=\frac{-2x-y}{x+2y}$. The horizontal tangent is when $y'=0$, which occurs when $-2x-y=0$ or $x=\frac{-y}{2}$. After substituting this back into the equation of the curve, we get $\left(\frac{-y}{2}\right)^2+\left(\frac{-y}{2}\right)y+y^2=\frac{3y^2}{4}=27$. Since $L1$ must pass through the first quadrant, this means that it is the line $y=6$.
L2: We first compute that $y'=3x^2-20x+31$, which means that the slope of the tangent line at $x=5$ is $3(5)^2-20(5)+31=6$. When $x=5$, $y=0$, and so the equation for the tangent line is $y=6x-30$.
L3: We first compute that $y'=-4x^3+3x^2+2x+4$, which is not equal to 0 for any value of x in our given domain. We can note that $y'>0$ across the entire domain, which means that our curve is always increasing. Thus, the absolute minimum is when $x=-2$ and $y=-31$, and the absolute maximum is when $x=1$ and $y=2$. The equation of the line between these two points is $y=11x-9$.
To evaluate the area of our triangle, we need to first compute the points of intersection between each pair of lines. $L1$ and $L2$ intersect at the point $(6,6)$, $L1$ and $L2$ intersect at the point $(\frac{15}{11},6)$, and $L2$ and $L3$ intersect at the point $(\frac{-21}{5},\frac{-276}{5})$. The area of this triangle is $\frac{1}{2}\left(6-\frac{15}{11}\right)\left(6-\left(\frac{-276}{5}\right)\right)=\boxed{\frac{7803}{55}}$.

24. **Answer: A** $\left(\frac{77}{29\sqrt{485}}\right)$

Rohan starts at point A, which is $(-2,53)$. Based on the description for Point B, it must be an inflection point where the curve goes from being concave up to concave down. We can compute that $y''=-3x^2-9x+30=-3(x+5)(x-2)$. The value of x that satisfies the condition we are looking for is $x=2$, and thus, point B can be found to be $(2,101)$. Point C, the origin, is $(0,0)$. We can evaluate the value of $\sin(\angle BAC)$ via the cross product of $\overrightarrow{AB}=\langle 4,48,0\rangle$ and $\overrightarrow{AC}=\langle 2,-53,0\rangle$. $\langle 4,48,0\rangle\times\langle 2,-53,0\rangle=\langle 0,0,-308\rangle$. We know that $\Big|\langle 0,0,-308\rangle\Big|=\Big|\langle 4,48,0\rangle\Big|\Big|\langle 2,-53,0\rangle\Big|\sin(\angle BAC)$, which means that $\sin(\angle BAC)=\frac{308}{(\sqrt{2813})(4\sqrt{145})}=\boxed{\frac{77}{29\sqrt{485}}}$.

25. **Answer: B** $\left(\frac{e^{1-e-e^e}}{1+2e}\right)$

Since $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$, we have that $\frac{d}{dx}[f^{-1}(e^{e^e})] = \frac{1}{f'(f^{-1}(e^{e^e}))} = \frac{1}{f'(e)}$. We can use logarithmic differentiation to evaluate $f'(x)$. If we let $f(x) = y$, then we have that $y = x^{x^x}$. This equation can then be converted to the form $\ln(\ln y) = \ln[x^x \cdot \ln x] = x\ln x + \ln(\ln x)$. Using the $u-$substitution $u = \ln y$, we can then differentiate this equation to get $\frac{1}{y\ln y}\cdot\frac{dy}{dx} = \ln x + \frac{1}{x\ln x} + 1$. We can further simplify this to get $\frac{dy}{dx} = y\ln y[\ln x + \frac{1}{x\ln x} + 1] = x^{x^x}\cdot\ln(x^{x^x})[\ln x + \frac{1}{x\ln x} + 1] = x^{x^x+x}\left([\ln(x)]^2 + \ln x + \frac{1}{x}\right)$. Therefore, $\frac{1}{f'(e)} = \frac{1}{e^{e^e+e}[2+\frac{1}{e}]} = \boxed{\frac{e^{1-e-e^e}}{1+2e}}$.

26. **Answer: D** (π)

This integral can be simplified using the substitution $u = \frac{\pi}{2} - v$.

$$\int_0^{\frac{\pi}{2}}\left(\frac{4}{1+\tan^{\pi^2e^{30}}(v)}\right)dv = \int_{\frac{\pi}{2}}^{0}\left(\frac{-4}{1+\tan^{\pi^2e^{30}}(\frac{\pi}{2}-u)}\right)du = \int_0^{\frac{\pi}{2}}\left(\frac{4}{1+\cot^{\pi^2e^{30}}(u)}\right)du = \int_0^{\frac{\pi}{2}}\left(\frac{4\tan^{\pi^2e^{30}}(u)}{1+\tan^{\pi^2e^{30}}(u)}\right)du.$$

$$2\int_0^{\frac{\pi}{2}}\left(\frac{4}{1+\tan^{\pi^2e^{30}}(u)}\right)du = \int_0^{\frac{\pi}{2}}\left(\frac{4+4\tan^{\pi^2e^{30}}(u)}{1+\tan^{\pi^2e^{30}}(u)}\right)du = \int_0^{\frac{\pi}{2}}(4)\,du = 2\pi.$$

Thus,

$$\int_0^{\frac{\pi}{2}}\left(\frac{4}{1+\tan^{\pi^2e^{30}}(v)}\right)dv = \frac{2\pi}{2} = \boxed{\pi}.$$

27. **Answer: A** $\left(\frac{1}{6}\right)$

We can evaluate this limit by repeatedly using L'Hospital's rule.

$$\lim_{x\to-1}\left(\ln\left(\left(\frac{\sin(x+1)}{x+1}\right)^{\frac{1}{2(\cos(x+1)-1)}}\right)\right) = \lim_{x\to0}\left(\ln\left(\left(\frac{\sin(x)}{x}\right)^{\frac{1}{2(\cos(x)-1)}}\right)\right) = \lim_{x\to0}\left(\frac{\ln(\sin(x))-\ln(x)}{2(\cos(x)-1)}\right) =$$

$$\lim_{x\to0}\frac{x\cos(x)-\sin(x)}{-2x\sin^2(x)} = \lim_{x\to0}\left(\frac{-x\sin(x)}{-2\sin^2(x)-2x\sin(2x)}\right) = \lim_{x\to0}\left(\frac{-x}{-2\sin(x)-4x\cos(x)}\right) =$$

$$\lim_{x\to0}\left(\frac{-1}{-6\cos(x)+4x\sin(x)}\right) = \frac{1}{6}.$$

Thus,

$$\lim_{x\to-1}\left(\frac{\sin(x+1)}{x+1}\right)^{\frac{1}{2(\cos(x+1)-1)}} = \boxed{e^{\frac{1}{6}}}.$$

28. **Answer: C** $\left(5\sqrt{13}-1\right)$

We can first compute the total volume of Tanusri's balloon using the second theorem of Pappus. Our volume is equal to the product of the area of the hexagon and the distance traveled by the centroid of the hexagon during the rotation. The area of the hexagon is $\frac{3s^2\sqrt{3}}{2} = \frac{3(6)^2\sqrt{3}}{2} = 54\sqrt{3}$ square feet. The centroid of the hexagon is its center, and the radius of the circular path that the centroid travels during our rotation is the apothem. The apothem of the hexagon has a length of $6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$ feet. The distance traveled by the centroid of the hexagon during the rotation is therefore $(2\pi)(3\sqrt{3}) = 6\pi\sqrt{3}$ feet, and the volume of the balloon is therefore $(54\sqrt{3}$ square feet$)(6\pi\sqrt{3}$ feet$) = 972\pi$ cubic feet. We can treat the rest of this problem like a kinematics question pertaining to distance, acceleration, and velocity. Let $t =$ time. $972\pi = 6\pi t + \frac{6\pi t^2}{2}$, and solving using the quadratic formula, we get that $t = \boxed{5\sqrt{13}-1}$ seconds.

29. **Answer: E** (1)

We can simplify these three integrals using u-substitutions.

$$\int_2^4 f'(2x)f(2x)\,dx = \frac{1}{2}\int_{f(4)}^{f(8)} u\,du = \frac{1}{2}\left(\frac{u^2}{2}\right)\Bigg|_{f(4)}^{f(8)} = \frac{[f(8)]^2-[f(4)]^2}{4} = 4. \text{ Thus, } [f(8)]^2-[f(4)]^2 = 16.$$

$$\int_8^0 f'(x)f(x)\,dx = \int_{f(8)}^{f(0)} u\,du = \left(\frac{u^2}{2}\right)\Bigg|_{f(8)}^{f(0)} = \frac{[f(0)]^2-[f(8)]^2}{2} = 0. \text{ Thus, } [f(0)]^2-[f(8)]^2 = 0.$$

$$\int_{\frac{1}{2}}^0 f'(8x)f(8x)\,dx = \frac{1}{8}\int_{f(4)}^{f(0)} u\,du = \frac{1}{8}\left(\frac{u^2}{2}\right)\Bigg|_{f(4)}^{f(0)} = \frac{[f(0)]^2-[f(4)]^2}{16}.$$

We can add the equations $[f(8)]^2-[f(4)]^2 = 16$ and $[f(0)]^2-[f(8)]^2 = 0$ together to find that $[f(0)]^2-[f(4)]^2 = 16$. Thus $\frac{[f(0)]^2-[f(4)]^2}{16} = \frac{16}{16} = \boxed{1}$.

30. **Answer: B** $\left(\frac{6n-24}{n(1-n)^4}\right)$

$$\sum_{i=0}^{\infty} n^{i-3}i(-3i-6)(i-1) = -3\sum_{i=0}^{\infty} n^{i-3}i(i+2)(i-1) = -3\left(\sum_{i=0}^{\infty} n^{i-3}i(i-1)(i-2) + 4\sum_{i=0}^{\infty} n^{i-3}i(i-1)\right) =$$

$$-3\left(\sum_{i=0}^{\infty} n^{i-3}i(i-1)(i-2) + \frac{4}{n}\sum_{i=0}^{\infty} n^{i-2}i(i-1)\right) = -3\left(\frac{d^3}{dn^3}\sum_{i=0}^{\infty} n^i + \frac{4}{n}\frac{d^2}{dn^2}\sum_{i=0}^{\infty} n^i\right) =$$

$$-3\left(\frac{d^3}{dn^3}\frac{1}{1-n} + \frac{4}{n}\frac{d^2}{dn^2}\frac{1}{1-n}\right) = -3\left(\frac{6}{(1-n)^4} + \frac{4}{n}\frac{2}{(1-n)^3}\right) = \boxed{\frac{6n-24}{n(1-n)^4}}.$$