For all questions, the answer choice **(E)** NOTA *means that none of the given answers is correct. Good Luck!*

(**1 − 5**) Let's get started with some limits and derivatives!

1. If $f(x) = (x+1)(x+2)$, find $f'(0)$.

(A) 2 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** NOTA

2. Suppose that [left =]align*f(x) =x$^4 - x^3 - 3x^2 + x + 2$
$g(x) = -x^2 - 2x - 1$ If $h(x)$ is a continuous function such that $f(x) \geq h(x) \geq g(x)$ for all x, find $\lim_{x \to -1} h(x)$.

(A) -1 **(B)** 0 **(C)** 1 **(D)** need more info **(E)** NOTA

3. Find $\lim_{x \to \infty} \frac{\sqrt{x^2 + 2x + 2}}{x}$.

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** NOTA

4. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find $\lim_{x \to 5^-} (x \cdot \lfloor x \rfloor)$.

(A) 16 **(B)** 20 **(C)** 25 **(D)** does not exist **(E)** NOTA

5. Given that $y = xe^{xy}$, find $\frac{dy}{dx}$ where defined.

(A) $\frac{y + xy^2}{x - x^2y}$ **(B)** $\frac{y + xy^2}{x + x^2y}$ **(C)** $\frac{x + xy^2}{x - x^2y}$ **(D)** $\frac{x + xy^2}{x + x^2y}$ **(E)** NOTA

($\mathbf{6-10}$) Recall from Precalculus that $\sin^2(x)+\cos^2(x)=1$. Let's put this fact to use!

6. Where defined, find $\dfrac{d}{dx}\left(\dfrac{\cos(x)}{\sin(x)}\right)$.

(A) $\sec^2(x)$ **(B)** $-\sec^2(x)$ **(C)** $\csc^2(x)$ **(D)** $-\csc^2(x)$ **(E)** NOTA

7. What is the area of the region in the xy-plane bounded by the graphs of $y=\sin^2(x)$, $y=-\cos^2(x)$, $x=-\pi/4$, and $x=\pi/4$?

(A) $\pi/8$ **(B)** $\pi/4$ **(C)** $\pi/2$ **(D)** π **(E)** NOTA

8. At time $t=0$ seconds, a particle starts at the point $(3,0)$. If the position of the particle after t seconds have elapsed is $(3\cos(t),3\sin(t))$, what is the distance traveled by the particle in the first 3π seconds?

(A) 3π **(B)** 6π **(C)** 9π **(D)** 12π **(E)** NOTA

9. Compute $\left(\int_0^1 x\sin(x)\,dx\right)^2$.

(A) $1-\sin(2)$ **(B)** $1-2\sin(1)$ **(C)** $1-2\cos(1)$ **(D)** 1 **(E)** NOTA

10. Compute $\int_0^{\pi/2}\sin^2(x)\,dx$.

(A) $1/2$ **(B)** $\pi/4$ **(C)** 1 **(D)** $\pi/2$ **(E)** NOTA

(**11 − 15**) Let's explore some ways in which Calculus can be applied to real-world problems!

11. You're running a widget factory at which it costs $\$(100 - 10w + w^2)$ to make w widgets. If you can sell each widget for \$50, how many widgets should you make to maximize your total profit?

(A) 10 **(B)** 20 **(C)** 30 **(D)** 40 **(E)** NOTA

12. Farmer Alex has 140 meters of fencing to build a rectangular grazing area for his cow. He is building the grazing area along a very long, perfectly straight river, so he can use the river to form one boundary of the rectangle. To maximize the grazing area, what should be the length of the longest side of the rectangle (in meters)?

(A) 35 **(B)** $^{140}/_3$ **(C)** 70 **(D)** 75 **(E)** NOTA

13. The quantum mechanical model of an atom describes the position of electrons as a probability distribution. Consider $P(r) = \dfrac{r^2 \cdot e^{-r}}{2}$ where $r \geq 0$ is the distance from the nucleus. What distance r maximizes $P(r)$?

(A) 1 **(B)** $\ln(4)$ **(C)** 2 **(D)** e **(E)** NOTA

14. Your ice cream has melted and perfectly fills up your giant ice cream cone, which has radius 10 inches and height 20 inches. Unfortunately, there is a small hole in the bottom (apex) of the cone, and the ice cream is leaking out at a constant rate of 100π cubic inches per minute. At what rate (in inches per minute) is the height of your ice cream decreasing when the height of your ice cream is 10 inches?

(A) $^1/_2$ **(B)** 1 **(C)** 2 **(D)** 4 **(E)** NOTA

15. Newton's Law of Cooling says that the rate at which an object is cooling is directly proportional to the difference in temperature between an object and its surroundings. A bottle of water (initially 70°F) is placed in a freezer which has a constant temperature of 30°F, and after 30 minutes the bottle of water is 50°F. Using Newton's Law of Cooling, find the temperature of the bottle after another 30 minutes have passed.

Hint: The definition of Newton's Law of Cooling gives us $\dfrac{d\theta}{dt} = k\left(\theta - 30°\text{F}\right)$, where $\theta(0) = 70°\text{F}$ and $\theta(30) = 50°\text{F}$. We can rewrite this as $\frac{d\theta}{\theta - 30} = k\,dt$. Integrate the left-side with respect to θ and the right-side with respect to t.

(A) 32°F **(B)** 35°F **(C)** 36°F **(D)** 40°F **(E)** NOTA

(**16** – **20**) Let's use Calculus to find some areas!

---

16. Using a left-hand Riemann rectangular approximation with 4 equal subdivisions, approximate $\int_0^\pi \sin(x)\,dx$. You may find the following values useful:

$$\sin(0) = 0, \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \qquad \sin\left(\frac{\pi}{2}\right) = 1, \qquad \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}, \qquad \sin(\pi) = 0.$$

(A) $\frac{\sqrt{2}}{4} \cdot \pi$ **(B)** $\frac{2+\sqrt{2}}{8} \cdot \pi$ **(C)** $\pi/2$ **(D)** $\frac{1+\sqrt{2}}{4} \cdot \pi$ **(E)** NOTA

---

17. Using a right-hand Riemann rectangular approximation with 4 equal subdivisions, approximate $\int_0^\pi \sin(x)\,dx$.

(A) $\frac{\sqrt{2}}{4} \cdot \pi$ **(B)** $\frac{2+\sqrt{2}}{8} \cdot \pi$ **(C)** $\pi/2$ **(D)** $\frac{1+\sqrt{2}}{4} \cdot \pi$ **(E)** NOTA

---

18. Using a trapezoidal approximation with 4 equal subdivisions, approximate $\int_0^\pi \sin(x)\,dx$.

(A) $\frac{\sqrt{2}}{4} \cdot \pi$ **(B)** $\frac{2+\sqrt{2}}{8} \cdot \pi$ **(C)** $\pi/2$ **(D)** $\frac{1+\sqrt{2}}{4} \cdot \pi$ **(E)** NOTA

---

19. Find the area of the region in the xy-plane bounded by the graphs of $y = 0$, $y = \sin(x)$, $x = 0$ and $x = \pi$.

(A) 1 **(B)** $\pi/2$ **(C)** 2 **(D)** π **(E)** NOTA

---

20. Find area of the region in the xy-plane bounded by the graphs of $y = \sin^2(x)$ and $y = \cos^2(x)$ for $-\pi/4 \le x \le \pi/4$.

(A) $1/2$ **(B)** 1 **(C)** 2 **(D)** 4 **(E)** NOTA

(21 – 25) Let's do some sequences/series. There are very helpful hints in each of the questions!

---

21. What constant β minimizes $\prod_{n=1}^{2015}(\beta e^{-n\beta})$? **Hint:** For a quick solution, take the natural log of the expression.

(A) $\frac{1}{2014!}$ **(B)** $\frac{1}{1008}$ **(C)** 1008 **(D)** 2014! **(E)** NOTA

---

22. Compute

$$\sum_{k=1}^{2015} k \cdot \binom{2015}{k} = 1 \cdot \binom{2015}{1} + 2 \cdot \binom{2015}{2} + \ldots + 2015 \cdot \binom{2015}{2015}.$$

Hint: Relate this to a derivative of $(x+1)^n$. You should recall the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

(A) 2^{2015} **(B)** $2015 \cdot 2^{2014}$ **(C)** 3^{2015} **(D)** $2015 \cdot 3^{2014}$ **(E)** NOTA

---

23. Consider the sequence $a_1, a_2, \ldots$ where $a_n = \sin^2\left(\pi \cdot \sqrt{n^2+n}\right)$. What does the sequence converge to?
Hint: Use the "completing the square" technique from Algebra on n^2+n.

(A) 0 **(B)** $1/2$ **(C)** 1 **(D)** does not converge **(E)** NOTA

---

24. $\sum_{n=1}^{\infty} \frac{2n-1}{(2n)!} = A + B \cdot e + C \cdot e^{-1}$ where A, B, and C are rational numbers. Find $A+B+C$. **Hint:** $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$.

(A) 0 **(B)** $1/2$ **(C)** 1 **(D)** $3/2$ **(E)** NOTA

---

25. You may know that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{and} \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan(x) = \sum_{k=0}^{\infty} c_k x^k$ where $c_0, c_1, c_2, \ldots$ are constants. Find c_5. **Hint:** $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

(A) $1/8$ **(B)** $7/60$ **(C)** $2/15$ **(D)** $13/60$ **(E)** NOTA

(**26** – **30**) Let's finish up with some assorted interesting problems. Good luck!

26. Let $f(x)$ be a continuous function such that $f'(x) = -f'(-x)$ for all $x \neq 0$. Is it necessarily true that $f'(0) = 0$?
(A) Yes **(B)** No

27. Compute $\int_1^4 \frac{1}{x + \sqrt{x}}\,dx$.
(A) $\ln(3/2)$ **(B)** $\sqrt{2} - 1$ **(C)** $\frac{\pi}{4}$ **(D)** $\ln(9/4)$ **(E)** NOTA

28. A cylinder has surface area 2015 square inches (including the circular bases). In order to maximize its volume, what should be the ratio of its radius to its height?
(A) 1 : 2 **(B)** 2 : 3 **(C)** 3 : 2 **(D)** 2 : 1 **(E)** NOTA

29. How many values $x = c$ satisfy Rolle's Theorem for the function $f(x) = x\sin(x)$ on the interval $x \in [-10, 10]$?
(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** NOTA

30. Let $f(x) = x^6 - 6x^2 + 6x - 7$, and note that $f'(x) = 0$ has exactly three real number solutions: a, b, and c. If $g(x)$ is a polynomial of degree two such that $g(a) = f(a)$, $g(b) = f(b)$, and $g(c) = f(c)$, find $g(2)$.

Hint: Since $f'(x) = 0$ for $x = a, b, c$, we know that $g(x) = f(x) + f'(x) \cdot h(x)$ at $x = a, b, c$ for any $h(x)$.
(A) -13 **(B)** -11 **(C)** -7 **(D)** 45 **(E)** NOTA