*For all questions, answer choice (E) NOTA means that none of the given answers is correct. Good Luck!*

1. Evaluate $\lim_{x\to 5} \dfrac{x-5}{\sqrt{x}-\sqrt{5}}$.

(A) DNE (B) $\sqrt{5}$ (C) $2\sqrt{5}$ (D) 0 (E) NOTA

2. Evaluate $\int_0^{\pi} \sin x\, dx$.

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) NOTA

3. Find $\frac{d}{dx}(\sin^2 x + \cos^2 x)$.

(A) 1 (B) 0 (C) $2\sin 2x$ (D) $\sin 2x$ (E) NOTA

4. Let $f(x)$ be a differentiable function on the domain of real numbers, such that

$$f(x) = \begin{cases} -x^2 + x + 1, & \text{if } x < 0 \\ e^{-ax}, & \text{if } x \geq 0 \end{cases},$$

where $a$ is a real number. What is $a$?

(A) $-1$ (B) 0 (C) $\frac{1}{2}$ (D) 1 (E) NOTA

5. Let $a$ be the slope of the tangent line to the function $f(x) = \sin(x^2) - \cos(x^2)$ at $x = \frac{\sqrt{\pi}}{2}$. What is $a^2$?

(A) $\sqrt{2\pi}$ (B) $2\pi$ (C) $\sqrt{\pi}$ (D) $\pi$ (E) NOTA

6. Evaluate $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$.

(A) DNE (B) 1 (C) $\frac{5}{3}$ (D) $\frac{3}{5}$ (E) NOTA

7. Panpan and Lisa are enjoying a romantic candlelit dinner. The candle is cylindrically-shaped with a radius of 2 cm and decreases in volume at a constant rate of 1 $\text{cm}^3$/min. What is the rate of decrease of the surface area of the candle in $\text{cm}^2$/min?

(A) 1 (B) $\frac{2}{\pi}$ (C) $\frac{1}{\pi}$ (D) 2 (E) NOTA

8. If $f(x) = \sin^3 x + \cos^3 x$, compute $f'(x)$ at $x = \frac{\pi}{2}$.

(A) 0 (B) 1 (C) 3 (D) 6 (E) NOTA

9. Compute the sum of the critical points of the function $f(x) = x^4 - 28x^3 - 6x^2 + 4x + 7$.

(A) $-21$ (B) $-7$ (C) 21 (D) 28 (E) NOTA

10. The inflection points of the function $f(x) = e^{-x^2}$ share a common $y$-coordinate. What is this value?

(A) $\frac{\sqrt{2}}{2}$ (B) 1 (C) $e^{-\frac{1}{2}}$ (D) $e^{-\frac{\sqrt{2}}{2}}$ (E) NOTA

11. Use the Midpoint rule on $[0, 3]$ with three equal subintervals to approximate the value of $\int_0^3 \cos(\pi x^2)\,dx$.

(A) $-\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) 1 (D) $\frac{3\sqrt{2}}{2}$ (E) NOTA

12. A particle is moving on the $x$-axis, with a position function of $p(t) = t^2 - t$, where $t$ represents time. What is the average value of the particle's velocity on the interval $[1, 2]$?

(A) 0 (B) $\frac{5}{6}$ (C) 1 (D) 2 (E) NOTA

13. There is a family of functions that are the solution to $\int -\frac{dx}{2}$. Consider the parabola $y = x^2$ and the line $y = 2x$. One of the functions in the aforementioned function family intersects the parabola and the line between the origin and $x = 2$. What is the greatest distance between the point of intersection with the line and the point of intersection with the parabola?

(A) 0 (B) $\frac{\sqrt{5}}{5}$ (C) $\frac{2\sqrt{5}}{5}$ (D) $\frac{1}{2}$ (E) NOTA

14. Jenny and Siddarth enjoy lying together under the light of a starry night sky, gazing up at the heavens in the midst of a calm meadow. Jenny always spots Polaris, the north pole star. Meanwhile, Siddarth likes to first spot Sirius, the brightest star in the night sky. They map the appearance of the night sky in the form of a polar graph over a period of time, with Polaris as the origin, and Siddarth notes that Sirius follows a path of the form $r = \frac{1 + \sqrt{\sin\theta}}{\sin\theta}$. What is the closest distance that Sirius is from Polaris on the map?

(A) $\frac{\sqrt{3}}{2}$ (B) 2 (C) 1 (D) $\frac{1}{2}$ (E) NOTA

15. Compute $\int_0^1 \frac{e^x}{e^{2x} + 1}\,dx$.

(A) $\arctan e - \frac{\pi}{4}$ (B) $e - 1$ (C) $\ln\frac{\pi}{4}$ (D) $\arctan e$ (E) NOTA

16. Li'l Rami is playing as a lone striker in the Nicaraguan national football team in the 2018 World Cup, and he needs to make one more goal for them to qualify for the finals, and the goalkeeper is distracted. He kicks the ball at an angle of 30° with a velocity of 60 m/s. Assuming that acceleration due to gravity is 10 m/s$^2$, what is the maximum distance in meters that Li'l Rami can be from the goal in order to kick the ball to or past the goal line? (Note: Assume that the ball does not roll or bounce after the first impact)

(A) $60\sqrt{3}$ (B) 180 (C) $180\sqrt{3}$ (D) 240 (E) NOTA

17. Consider the region bound between the functions $y = x^2 + 1$ and $y = -x^2 + 2$; compute the volume of the solid generated by rotating this region around the $x$-axis.

(A) $2\pi\sqrt{2}$ (B) $2\sqrt{2}$ (C) $\frac{2\pi\sqrt{2}}{3}$ (D) $\frac{2\sqrt{2}}{3}$ (E) NOTA

18. A triangular prism with an equilateral base is growing in volume at a constant rate of $20\sqrt{3}$ units$^3$/sec, and the height of the prism is growing at a constant rate of 4 units/sec. What is the rate of change of the side length of the base when the height is 5 units and the side length is 2 units?

(A) $\frac{18}{5}$ (B) 16 (C) $\frac{16\sqrt{3}}{15}$ (D) $\frac{16\sqrt{3}}{3}$ (E) NOTA

19. Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$.

(A) Diverges (B) $e$ (C) $\frac{e^2+1}{2e}$ (D) $\frac{e^2-1}{2e}$ (E) NOTA

20. Calculus can actually help quite a bit with olympiad inequalities, particularly convexity/concavity. Think about when the minima and maxima of a convex function occur, and use this to find the maximum of the following expression, for $0 \leq x, y, z \leq 1$:

$$\frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1} + (1-x)(1-y)(1-z).$$

(Hint: First, notice how it's symmetric in terms of $x, y, z$. Now let $f(x)$ be this expression, with $x$ as the variable and $y$ and $z$ being arbitrary constants. What's the sign of the second derivative of $f(x)$?)

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3 (E) NOTA

**Tricks tend to be very useful in calculus when it comes to computations. For questions 21-24, you will be provided a hint to a trick that will make the integrals easier to evaluate. Some of these tricks are simpler than others, but they can prove to be quite useful in evaluating integrals that otherwise look pretty tricky.**

21. Sometimes making substitutions for functions can simplify an integral's expresion, and makes it easier to see how to integrate it. Try this for evaluating the following antiderivative, the answer to which is the answer to this question:

$$\int \frac{x^4}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^4}{4!}}\, dx.$$

(Hint: try clearing the denominator. What's the derivative of the bottom function?)

(A) $-\ln(x^4 + 4x^3 + 12x^2 + 24x + 24) + C$
(B) $-24\ln(x^4 + 4x^3 + 12x^2 + 24x + 24) + C$
(C) $x - \ln(x^4 + 4x^3 + 12x^2 + 24x + 24) + C$
(D) $24x - 24\ln(x^4 + 4x^3 + 12x^2 + 24x + 24) + C$
(E) NOTA

22. Substitution of a new variable can also sometimes make integrals either easier to evaluate, or sometimes reveal properties about them that make them easier to evaluate. If you took last year's calculus individual test (either via practice or during the real thing), you'll notice that I like having some trigonometric integrals involving symmetry (e.g. substituting $x$ for $\frac{\pi}{2} - u$ yielded the same value, but a different integral – the two integrals added up to something quite nice). This next integral doesn't use the exact same concept, but try to keep the idea in mind; the value of the following integral is the answer to this question:

$$\int_0^{\infty} \frac{\ln x}{x^2+1}\, dx.$$

(A) $\frac{\pi}{2}\ln 2$ (B) $\frac{1}{2}$ (C) 1 (D) $\pi \ln 2$ (E) NOTA

23. The tangent half-angle substitution (or known as the Weierstrass substitution to some) is a very useful technique; according to Spivak, "the world's sneakiest substitution is undoubtedly" this substitution[1]. It becomes very useful for evaluating trigonometric integrals, as it turns them strictly into integrals of rational functions. The substitution involves substituting $\tan\frac{x}{2}$ for the new variable $t$ (now try to find $\sin x$ and $\cos x$ in terms of $t$! Hint: remember that $\cos^2 x = \frac{1}{\sec^2 x}$). Now use this substitution to evaluate the following integral, the value of which is the answer to this question:

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{1+2\sin x}\, dx.$$

(A) $\frac{\sqrt{3}}{3}\ln\left(\frac{\sqrt{3}+1}{2}\right)$ (B) $1 - 2\ln\left(\frac{\sqrt{3}+1}{2}\right)$ (C) $\frac{\sqrt{3}}{3}\ln(\sqrt{3}-1)$ (D) $\frac{\sqrt{3}}{3}\ln(\sqrt{3}+1)$ (E) NOTA

[1] Michael Spivak, *Calculus.* Cambridge University Press, Cambridge, United Kingdom, 2006. p. 382-383.

24. This question involves one of Richard Feynman's favorite tools from his toolbox, one not often taught in the classroom. Aptly called "differentiation under the integral sign," a special case of it, known as the Leibniz integral rule, states the following: given the two-variable function $f(a, x)$, let $F(a) = \int_{x_0}^{x_1} f(a, x)\, dx$. Then

$$\frac{d}{da}F(a) = \frac{d}{dx}\left(\int_{x_0}^{x_1} f(a,x)\,dx\right) = \int_{x_0}^{x_1} \frac{\partial}{\partial a} f(a,x)\,dx,$$

where $\frac{\partial}{\partial a} f(a, x)$ refers to the result if you treat $x$ like a constant and differentiate with respect to $a$. Use this to evaluate the following integral, the value of which is the answer to this question[2]:

$$\int_0^\infty \frac{\arctan \frac{2014}{x} - \arctan \frac{1}{x}}{x}\,dx.$$

(Hint: introduce a new variable $a$. Perhaps substitute a number for $a$.)

(A) $\pi \ln 2014$ (B) $\frac{\pi}{2} \ln 2014$ (C) $\frac{\pi}{4} \ln 1012$ (D) $\frac{\pi}{2} \ln 1012$ (E) NOTA

25. Evaluate $\int \arctan\left(\frac{1-x}{x}\right) dx$.

(A) $x \arctan\left(\frac{1-x}{x}\right) + \frac{\ln(2x^2 - 2x + 1)}{4} + C$

(B) $x \arctan\left(\frac{1-x}{x}\right) + \frac{\ln(2x^2 - 2x + 1)}{2} - \frac{\arctan(2x-1)}{2} + C$

(C) $x \arctan\left(\frac{1-x}{x}\right) - \frac{\ln(2x^2 - 2x + 1)}{4} - \frac{\arctan(2x-1)}{2} + C$

(D) $x \arctan\left(\frac{1-x}{x}\right) + \frac{\ln(2x^2 - 2x + 1)}{4} + \frac{\arctan(2x-1)}{2} + C$

(E) NOTA

26. In analytic number theory, a useful identity (known as Abel's identity[3]) can be stated as the following: for a function $a(n)$ defined on the positive integers, let

$$A(x) = \sum_{n \le x} a(n),$$

where $A(x) = 0$ if $x < 1$. Let $f(x)$ be a function with a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Additionally, it is known that there exist real numbers $c_1, c_2$ such that

$$\ln x + c_1 < \sum_{n \le x} \frac{\ln p_n}{p_n} < \ln x + c_2,$$

where $p_n$ denotes the $n$th prime. Using these facts, which of the following series converge? (Hint: Consider $a(n)$ where $a(n)$ is one type of term if $n$ is prime, and 0 otherwise)

I. $\sum_{n=1}^{\infty} \frac{1}{p_n}$ II. $\sum_{n=1}^{\infty} \frac{1}{p_n \ln p_n}$ III. $\sum_{n=1}^{\infty} \frac{\ln p_n}{p_n^2}$

(A) I, II (B) I, II, III (C) II only (D) III only (E) NOTA

[2] For those with additional calculus experience, there is another multivariable way to solve this integral. Try to find it after the test! Unless, of course, you already found it.

[3] If this looks like integration by parts, it's because it pretty much is. It's a consequence (read: can be proved by) the process of integration by parts for a generalized version of the Riemann integral, called the Riemann-Stieltjes integral.

27. Telescoping can often simplify integrals, but sometimes compressing them via telescoping makes it harder to see the solution, especially if the original terms are individually easier to integrate. Try to think about this when you evaluate the following integral, the value of which is the answer to this question:

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x \sin \dfrac{5x}{2}}{\sin \dfrac{x}{2}}\, dx$$

(A) $\frac{38}{15}$ (B) $\frac{28}{15}$ (C) $\frac{316}{105}$ (D) $\frac{43}{15}$ (E) NOTA

28. In chemistry, it is known that hydrogen ions ($H^+$) and ammonia ($NH_3$), a weak base, react in aqueous solution to form ammonium (${NH_4}^+$), a weak acid, in the following reaction: $H^+(aq) + NH_3(aq) \longrightarrow {NH_4}^+(aq)$. According to the Law of Mass Action, the rate at which ammonium is formed is jointly proportional to the amount of hydrogen ions and ammonia that have not yet reacted. At time $t = 0$, no ammonium has been formed. If the initial concentrations of hydrogen ions and ammonia are 0.25 M and 0.10 M, respectively, what is the concentration of ammonium at $t = 20$, given that the constant of proportionality is 1? Note: according to stoichiometry, if $x$ amount of ammonium has formed from an original $A$ and $B$ amounts of hydrogen ions and ammonia, then there will be $A - x$ and $B - x$ amounts of hydrogen ions and ammonia.

(A) $\frac{e^3 - 1}{10e^3 - 4}$ (B) $\frac{e^3 - 1}{4e^3 - 10}$ (C) $\frac{4e^3 - 1}{10e^3 - 1}$ (D) $\frac{4e^3 - 10}{10e^3 - 1}$ (E) NOTA

29. Evaluate

$$\lim_{n\to\infty} \frac{\left(1^{1^4} 2^{2^4} \cdots n^{n^4}\right)^{\frac{1}{n^5}}}{n^{\frac{1}{5}}}$$

(A) $-\frac{1}{25}$ (B) $e^{-\frac{1}{25}}$ (C) $-\frac{1}{16}$ (D) $e^{-\frac{1}{16}}$ (E) NOTA

30. Given an integrable periodic function $f(x)$ with a period of $2\pi$ and fixed positive integer $N$, a function $g(x)$ of the form $a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$ for some real numbers $a_0, a_1, \ldots, a_N, b_1, \ldots, b_N$ is denoted as an $(f, N)$-*periodic function of best fit* if $\int_{-\pi}^{\pi} (f(x) - g(x))^2\, dx$ is minimized over all possible values of $a_0, a_1, \ldots, a_N, b_1, \ldots, b_N$. Given a function $g(x)$ of the previously defined form, let $\mathcal{S}(g(x)) = a_0 + \cdots + a_N + b_1 + \cdots + b_N$. Let $f(x)$ be defined as

$$f(x) = \begin{cases} -1, & \text{if } \left\lfloor \dfrac{x}{\pi} \right\rfloor \text{ is odd} \\ 1, & \text{if } \left\lfloor \dfrac{x}{\pi} \right\rfloor \text{ is even} \end{cases},$$

where $\lfloor x \rfloor$ denotes the greatest integer function. Then calculate $\mathcal{S}(g(x))$, if $g(x)$ is the $(f, 7)$-periodic function of best fit.

(A) $\frac{352}{105}$ (B) $\frac{363}{35}$ (C) $\frac{216}{35}$ (D) $\frac{704}{105}$ (E) NOTA