For all questions, answer choice (E) NOTA means that none of the given answers is correct. Good Luck!

1. Evaluate: $\lim_{x\to\infty} \dfrac{3x^2+7x+12}{2x^2+5x+2}$

   (A) $\frac{3}{2}$ (B) ∞ (C) 0 (D) $\frac{2}{3}$ (E) NOTA

2. Evaluate: $\int_0^6 x^2\,dx$.

   (A) 9 (B) 24 (C) 36 (D) 72 (E) NOTA

3. Write $\lim_{a\to 0} \dfrac{(x+a)^5-x^5}{a}$ in terms of x.

   (A) x^5 (B) $5x^4$ (C) 0 (D) x^4 (E) NOTA

4. If $f(x) = x + \frac{1}{x}$, compute $f''(3)$.

   (A) $\frac{1}{9}$ (B) $-\frac{2}{27}$ (C) $\frac{2}{27}$ (D) $-\frac{1}{27}$ (E) NOTA

5. Evaluate $\int_0^2 e^{2x} + \frac{1}{4}\,dx$.

   (A) $\frac{e^4}{2}$ (B) $\frac{2e^4-1}{4}$ (C) $\frac{e^2}{2}$ (D) $\frac{e^4+1}{2}$ (E) NOTA

6. Consider the function

$$f(x) = \frac{2x^2-3x-9}{x^2-x-6}$$

   Let A be the number of slant asymptotes, B be the number of horizontal and vertical asymptotes, and let C be the number of removable discontinuities. Evaluate BC^A.

   (A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

7. The cubic polynomial x^3-3x+4 has critical points at (x_1, y_1) and (x_2, y_2). What is $x_1x_2 + y_1y_2$?

   (A) 13 (B) -4 (C) 11 (D) 4 (E) NOTA

8. Consider a particle traveling on the x-axis. If the position of the particle on the x-axis at time t is given by the function $p(t) = 3t^2+5t+7$, then evaluate the average velocity of the particle on the time interval $[1, 5]$.

   (A) 23 (B) $\frac{41}{2}$ (C) 53 (D) 6 (E) NOTA

9. Approximate $\int_0^4 \sqrt{x}\,dx$ using a right-hand Riemann sum with 4 equal subdivisions.

   (A) $1+\sqrt{2}+\sqrt{3}$ (B) $3+\sqrt{2}+\sqrt{3}$ (C) $\frac{16}{3}$ (D) $\frac{18+4\sqrt{2}+6\sqrt{3}}{3}$ (E) NOTA

10. Let a, b be nonzero real numbers. Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} e^x & \text{if } x \le 1 \\ ax^2+b & \text{if } x > 1 \end{cases}$$

   If $f(x)$ is differentiable over all real numbers, find ab.

   (A) e (B) $\frac{e^2}{4}$ (C) $\frac{e}{2}$ (D) 1 (E) NOTA

11. A ball is rolled across a straight stretch of road at an initial velocity of 30 ft/s. The ball, however, slows down at a constant rate of deceleration of 3 ft/s^2 until it eventually reaches a full stop. How far away from the original starting point does the ball eventually reach?

(A) 165 (B) 200 (C) 300 (D) 450 (E) NOTA

12. Consider the functions $f(x) = \sin x$ and $g(x) = \sin 2x$. Evaluate the area of the regions bound by $x = \frac{\pi}{2}$ and the curves $y = f(x)$ and $y = g(x)$.

(A) $\frac{3}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$ (E) NOTA

13. Consider the curve defined by the equation $x^2 - xy + y^2 = 27$. Find the slope of the tangent line at the point $(3\sqrt{3}, \sqrt{3})$.

(A) $3 - 2\sqrt{3}$ (B) 5 (C) -5 (D) $9 - 3\sqrt{3}$ (E) NOTA

14. Let $f(x) = 2x^3 - 3x^2 + 6x + 1$. If $g(x) = f^{-1}(x)$, then compute $g'(6)$.

(A) 6 (B) 1 (C) $\frac{1}{18}$ (D) $\frac{1}{6}$ (E) NOTA

15. Kevin is blowing up a perfectly spherical balloon at a constant rate of 8 in^3/s. At some time t, the rate of change of the surface area of the balloon is 2 in^2/s. What is the rate of change of the radius at this moment in time? All answers are in terms of in/s.

(A) 8 (B) $\frac{1}{32\pi}$ (C) $\frac{1}{8\pi}$ (D) 2 (E) NOTA

16. Evaluate: $\lim\limits_{x\to 0^+} \dfrac{\int_0^{x^2} \sqrt{t}\cos t\, dt}{x^3}$.

(A) 1 (B) $\frac{1}{3}$ (C) DNE (D) $\frac{2}{3}$ (E) NOTA

17. Consider a solid figure A such that the base is a circle of radius 2 and the cross sections are equilateral triangles. What is the volume of A?

(A) $\frac{16\pi\sqrt{3}}{3}$ (B) $\frac{32\sqrt{3}}{3}$ (C) $\frac{64\sqrt{3}}{3}$ (D) $\frac{64\pi\sqrt{3}}{3}$ (E) NOTA

18. Compute the area of the region enclosed by the polar curve $r = \cos\left(\frac{\theta}{2}\right)$ and the x-axis for $0 \le \theta \le \pi$. (Hint: for $0 \le \theta \le \pi$, $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{8}$ (C) $\frac{\pi+1}{4}$ (D) $\frac{1}{2}$ (E) NOTA

19. Evaluate the limit: $\lim\limits_{n\to\infty} \sum_{i=1}^{3n} \frac{1}{n}\left(\frac{i^2}{n^2} + \sqrt{\frac{i}{n}}\right)$.

(A) 1 (B) $9 + 3\sqrt{3}$ (C) 11 (D) $9 + 2\sqrt{3}$ (E) NOTA

For questions 20-22, use the following information:
Hyun Jee loves cupcakes. However, she only enjoys eating the highest quality cupcakes, and this depends on the amount of frosting it has, as well as the shape. The Tasty factor for the cupcake with x amount of frosting is represented by $t(x) = x^2 e^{-(x-1)^2}$. Additionally, she only eats cupcakes that have a certain shape. The cupcake must be the figure obtained by rotating the following function about the x-axis from $x = 0$ to $x = 2$:

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \sqrt{x}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } 1 < x \leq 2 \\ 0, & \text{if } x > 2 \end{cases}$$

20. What is the volume of the cupcakes that Hyun Jee eats?

(A) 2π (B) $\frac{3\pi}{2}$ (C) $\frac{4\pi}{3}$ (D) $\frac{5\pi}{3}$ (E) NOTA

21. If Hyun Jee only eats cupcakes with the maximum possible Tasty factor, what amount of frosting must be used for Hyun Jee's cupcakes?

(A) 1 (B) $\frac{1+\sqrt{5}}{2}$ (C) $\frac{1-\sqrt{5}}{2}$ (D) 0 (E) NOTA

22. The rate at which Hyun Jee eats the cupcake at time t is represented by the function

$$C(t) = \frac{54}{t^2 + 6t + 18}$$

At what time t will Hyun Jee finish the cupcake? (Hint: use your answer from Problem 20)

(A) 3 (B) $3 + 3\sqrt{3}$ (C) $3\sqrt{3} - 3$ (D) 9 (E) NOTA

23. Let

$$f(k) = \sqrt[k]{\frac{25^k + 24^k + \cdots + 2^k + 1^k}{k}}$$

Compute $\lim_{k\to\infty} f(k)$. (Hint: Squeeze theorem. What is $\lim_{k\to\infty} \left(\frac{1}{k}\right)^{\frac{1}{k}}$?)

(A) 25 (B) 325 (C) 1 (D) 0 (E) NOTA

24. If p_n denotes the nth prime, then it can be shown that for $n \geq 1$ that

$$p_n < 12\left(n \ln n + n \ln \frac{12}{e}\right)$$

Given this information, determine which of the following convergence modes the infinite series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{p_k}$ satisfies:

I. The series is conditionally convergent. II. The series is absolutely convergent. III. The series is divergent.

(A) I, II (B) II (C) III (D) I (E) NOTA

25. Consider the integral $\int_0^1 \frac{1}{x} \ln\left(\frac{1}{1-x}\right) dx$. The value of this integral is equal to which of the following sums?

(A) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ (B) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (C) $\sum_{k=1}^{\infty} \frac{1}{3^k}$ (D) $\sum_{k=1}^{\infty} \frac{(k-1)(-1)^{k-2}}{k^3}$ (E) NOTA

26. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt[3]{\tan x}}\,dx$$

(Hint: use symmetry)

(A) $\dfrac{\sqrt[3]{\pi}}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\sqrt[3]{\pi^2}}{2}$ (D) $\dfrac{\sqrt[3]{3}}{6}$ (E) NOTA

27. Let the gamma function, $\Gamma(x)$ be defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$$

It is a famous fact that the volume of an n-dimensional sphere of radius r is equal to $\dfrac{\pi^{\frac{n}{2}}r^n}{\Gamma\left(\frac{n}{2}+1\right)}$. Additionally, it is known that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

If $\Gamma\left(\dfrac{7}{2}\right) = \dfrac{15}{4}\Gamma\left(\dfrac{3}{2}\right)$, then evaluate the volume of the 5-dimensional unit sphere.

(A) $\dfrac{4\pi^{\frac{5}{2}}}{15}$ (B) $\dfrac{2\pi^3}{5}$ (C) $\dfrac{8\pi^2}{15}$ (D) $\dfrac{16\pi^3}{15}$ (E) NOTA

28. Let $f(x) = \arctan\left(e^x - 1\right)$. Evaluate the integral $\displaystyle\int_0^{\ln 2} f(x)\,dx$. (Hint: First consider the inverse function, and let $u = \dfrac{\pi}{4} - x$)

(A) $\dfrac{\pi}{8}\arctan(2)$ (B) $\dfrac{\pi}{4}\ln 2$ (C) $\dfrac{\ln 2}{4}$ (D) $\dfrac{\pi}{8}$ (E) NOTA

29. Evaluate the following limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{2}{n}}\left(1+\frac{2}{n}\right)^{\frac{3}{2n}}\cdots\left(1+\frac{k}{n}\right)^{\frac{k+1}{kn}}\cdots\left(1+\frac{n}{n}\right)^{\frac{n+1}{n^2}}$$

(A) 0 (B) $e^{\frac{1}{2}}$ (C) 1 (D) $\dfrac{4}{e}$ (E) NOTA

30. Let n be a positive integer and let x_1, x_2, x_3, x_4 be integers in the interval $[-n, n]$. Let $d(n)$ be the number of solutions to the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$. If $S(n) = \displaystyle\sum_{i=0}^{n^2} d(i)$, then evaluate $\displaystyle\lim_{n\to\infty}\frac{S(n)}{n^4}$.

(A) $\dfrac{\pi^2}{2}$ (B) 16 (C) e^4 (D) 0 (E) NOTA