2011 RICKARDS FALL INVITATIONAL
CALCULUS INDIVIDUAL TEST

The choice E. NOTA denotes None Of The Above. Good luck, and have fun!

SECTION I. LIMITS AND DERIVATIVES

1. Evaluate: $\displaystyle\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{2}}$

A. 0 B. $\frac{1}{4}$ C. 2 D. 4 E. NOTA

2. Evaluate: $\displaystyle\lim_{x\to\infty}\frac{x^6-10x^5+x^4-16x^3+1}{x^5\sin^2(x)+2x^4+3x^6+4}$

A. -5 B. $\frac{1}{3}$ C. 1 D. does not exist E. NOTA

3. If $f(x)=\begin{cases} ax^2+bx+c & \text{if } x\le -1 \\ bx^2+cx+a & \text{if } x>-1 \end{cases}$ is differentiable over the reals, find $f(-1)$. $(\{a,b,c\}\subset\mathbb{R})$

A. 0 B. a C. $2a$ D. $3a$ E. NOTA

4. Let $\min\{a,b\}=\begin{cases} a & \text{if } a\le b \\ b & \text{if } b\le a \end{cases}$. If $f(x)=\min\{x,1\}$, which of the following are real, finite numbers?

I. $f'(0)$ II. $f'(1)$ III. $f'(2)$

A. I only B. III only C. II and III only D. I, II and III E. NOTA

5. Compute $\dfrac{dy^2}{dx^2}$ at $x=3$ given that $x(t)=t^2+2t+3$ and $y(t)=t^3-2t+1$ for $t\neq 0$.

A. $-\frac{1}{2}$ B. $\frac{47}{256}$ C. 1 D. $\frac{47}{32}$ E. NOTA

SECTION II. APPLICATIONS

6. Find the minimum value of $f(x)=x+\dfrac{1}{2x}$ over the domain $x\in(0,\infty)$.

A. 0 B. $\frac{1}{2}$ C. $\frac{\sqrt{2}}{2}$ D. $\sqrt{2}$ E. NOTA

7. A car that is initially traveling at a rate of 56 ft/sec brakes at a constant rate (in other words, it has a constant negative acceleration), and comes to a stop in 4 seconds. What distance, in feet, does the car travel from the time it hits the brakes until it comes to a stop?

A. 56 B. 84 C. 112 D. 140 E. NOTA

8. Find all values of c that satisfy the Mean Value Theorem for Derivatives for $f(x)=\left|x^2-x+2\right|$ over the interval $x\in[0,4]$.

A. $\frac{1+\sqrt{5}}{2}$ B. 2 C. $\frac{5}{2}$ D. MVT doesn't apply E. NOTA

9. Eli is standing in the xy-plane at the point $(1, 0)$ and needs to get to Nuberri, which is located at the point $\left(-\frac{1}{2}, 0\right)$. He will get there by running $0 \le \theta \le \pi$ radians counterclockwise around the unit circle centered at the origin $(x^2 + y^2 = 1)$, and then walking straight to Nuberri. However, Eli can run twice as fast as he can walk. What angle θ minimizes the time it takes Eli to get to Nuberri?
A. 0　B. $\frac{2\pi}{3}$　C. $\frac{5\pi}{6}$　D. π　E. NOTA

10. A ladder that is 17 meters long is propped against a wall such that the base of the ladder is 5 meters from the wall. However, this is a collapsible ladder, so that at the same instant that the base of the ladder begins to slide away from the wall at 2 m/s, the length of the ladder starts to shrink at 1 m/s. Given that the ends of the ladder never leave either the wall or the ground, how fast is the upper end of the ladder moving down the wall in m/s two seconds after it starts to slip?
A. -1.5　B. 2.75　C. 2.8　D. 3.25　E. NOTA

<u>SECTION III. THE NUMBER e</u>

11. Pratik invests 1000 dollars in a bank with an interest rate of 4% per year that is compounded continuously. After 25 years, how much will this \$1000 be worth? Round to the nearest dollar.
A. \$2717　B. \$2718　C. \$2719　D. \$2720　E. NOTA

12. Evaluate: $\lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{\frac{1}{n}} \right)$
A. 0　B. $\frac{1}{e}$　C. 1　D. e　E. NOTA

13. Evaluate: $\lim_{n\to\infty} \left(\left(\frac{n}{n+3}\right)^{\frac{n}{3}} \right)$
A. $\frac{1}{e^9}$　B. $\frac{1}{e}$　C. e　D. e^9　E. NOTA

14. Let $f_n(x)$ and $g_n(x)$ be polynomials of degree n such that

$$f_n(n) = f_n'(n) = f_n''(n) = \cdots = f_n^{(n-1)}(n) = 0$$

and

$$g_n(n-1) = g_n'(n-1) = g_n''(n-1) = \cdots = g_n^{(n-1)}(n-1) = 0$$

Define $P\left(f_n(x)\right)$ as the product of the n roots (not necessarily distinct) of $f_n(x)$, and define $P\left(g_n(x)\right)$ equivalently. If $S_n = \dfrac{P\left(g_n(x)\right)}{P\left(f_n(x)\right)}$ for positive integer n, compute $\lim_{n\to\infty} S_n$.
A. 0　B. $\frac{1}{e}$　C. 1　D. e　E. NOTA

15. Evaluate: $\lim_{n\to\infty} \dfrac{1^n + 2^n + 3^n + \cdots + (n-1)^n}{n^n}$
A. 0　B. $\frac{1}{e}$　C. $\frac{1}{e-1}$　D. 1　E. NOTA

SECTION IV. INTEGRATION

16. Compute $\int_0^1 dx$.

A. -1 B. 0 C. 1 D. x E. NOTA

17. Compute $\int_0^1 \frac{x}{x^2+1}\, dx$.

A. $\ln\left(\frac{1}{2}\right)$ B. $\ln\left(\sqrt{2}\right)$ C. $\ln(2)$ D. $\ln(4)$ E. NOTA

18. Compute $\int_0^1 \sqrt{\frac{1}{(x+1)^4} - \frac{1}{(x+1)^5}}\, dx$.

A. $\frac{1}{8}$ B. $\frac{\sqrt{2}}{6}$ C. $\frac{\sqrt{2}}{4}$ D. $\frac{2}{3}$ E. NOTA

19. The integral $\int_0^1 \frac{dx}{x^2+2x+2}$ can be written as $\int_0^1 \frac{dx}{(x+1)^2+1}$. Using $x = \tan(\theta) - 1$, compute the value of this integral.

A. $\tan^{-1}(2)$ B. $\frac{2+\sin(2)}{4}$ C. $\frac{\pi}{4}$ D. 1 E. NOTA

20. For certain real values of p and q, there is a substitution that is suitable to evaluate $\int \frac{dx}{x^2+px+q}$ using the same method as in the previous question. For these values of p and q, the substitution is of the form $x = f(p,q) \cdot \tan(\theta) - \frac{p}{2}$. What is $f(p,q)$?

A. 1 B. $\frac{p}{q}$ C. $\sqrt{q - \frac{p}{2}}$ D. $\sqrt{q - \frac{p^2}{4}}$ E. NOTA

SECTION V. AREA AND VOLUME

21. Find the area bounded by the x-axis and the graphs of $x = 1$, $x = 6$, and $y = x^2$ in the coordinate plane.

A. $\frac{215}{3}$ B. $\frac{215\pi}{3}$ C. 1555 D. 1555π E. NOTA

22. Patrick has two shapes whose areas are changing in time: a square and an equilateral triangle. At any given time, if the side length of his equilateral triangle is equal to x units, then the rate of change of the area of his square is x units^2/sec. When the rate of change of the area of the square is equal to the rate of change of the area of the equilateral triangle, what is the rate of change of the perimeter of the equilateral triangle, in units/sec?

A. $\frac{\sqrt{3}}{2}$ B. $\frac{2\sqrt{3}}{3}$ C. $\frac{3\sqrt{3}}{2}$ D. $2\sqrt{3}$ E. NOTA

23. Bishoy's hair is in the shape of a hemisphere. The radius of his hair, $r(t)$, grows at a rate that is inversely proportional to the current radius of his hair. Initially (time $t = 0$ months), his hair has volume 18π and if he doesn't cut it for 6 months, the radius of his hair will be 8. After how many months (since $t = 0$) will Bishoy's hair have volume 144π? Round to the nearest month.

A. 2 B. 3 C. 4 D. 5 E. NOTA

24. The base of a solid is the region of points $\{(x, y, 0) \mid x^2 + y^2 = 9\}$. The cross sections of the solid by a plane perpendicular to the x-axis are squares which have two adjacent endpoints on the circle $x^2 + y^2 = 9$. Find the volume of this solid.

A. 36 B. 72 C. 108 D. 144 E. NOTA

25. A right circular cone with base radius of 1 and height of 2 is initially empty and filling from the base (the nose is down) at a rate of 1 cubic unit/minute. What is the rate of change of the height of the water in units/minute when the volume of the water is half the volume of the cone?

A. $\frac{\sqrt[3]{4}}{24\pi}$ B. $\frac{\sqrt[3]{4}}{12\pi}$ C. $\frac{\sqrt[3]{4}}{2\pi}$ D. $\frac{\sqrt[3]{4}}{\pi}$ E. NOTA

SECTION VI. ADVANCED TOPICS

26. Using a second-order Maclaurin polynomial for e^x, estimate $\displaystyle\int_0^1 \frac{x+1}{e^x}\,dx$.

A. $\ln(2)$ B. $\frac{2e-3}{e}$ C. $\ln\left(\frac{5}{2}\right)$ D. $\ln(4)$ E. NOTA

27. A differential equation of the form $\frac{dy}{dx} + p(x) \cdot y = q(x)$ is called *linear*. To solve this equation, multiply both sides of the equation by a special function, $u(x)$, called an *integrating factor*, to obtain $\frac{dy}{dx} \cdot u(x) + p(x) \cdot u(x) \cdot y = q(x) \cdot u(x)$. The goal is to choose $u(x)$ such that the left hand side of the equation is simply $\frac{d}{dx}(y \cdot u(x))$ so that the equation becomes $y \cdot u(x) = \int (q(x) \cdot u(x))\,dx$, and thus $y = \frac{\int (q(x) \cdot u(x))\,dx}{u(x)}$. Find the general form of $u(x)$ that has this special property.

A. e^x B. $e^{p'(x)}$ C. $e^{p(x)}$ D. $e^{\int p(x)\,dx}$ E. NOTA

28. Using polar coordinates and induction, it can be shown that the volume of an n-dimensional ball of radius r is

$$V_n(r) = \frac{(\pi r^2)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \qquad \text{for } n \geq 0$$

Find the sum of the volumes of the unit ball in every even dimension; that is, compute $\displaystyle\sum_{n=0}^{\infty} V_{2n}(1)$.

A. $e^{\sqrt{\pi}}$ B. e^{π} C. $e^{2\pi}$ D. e^{π^2} E. NOTA

29. Find the number of fifth degree polynomials $f(x)$ with positive integral coefficients that exist such that $f(0) = 1$, $f'(0) = 4$, and $f(1) = 16$.

A. 120 B. 210 C. 364 D. 1365 E. NOTA

30. The series representation of $\ln\left(\frac{1+x}{1-x}\right)$ for $x \in (-1, 1)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2x^{2n-1}}{2n-1}$. Given that $\log(e) \approx 0.4343$, find the number of digits, to the nearest fifty digits, in the base 10 expansion of 3^{2011}.

A. 850 B. 900 C. 950 D. 1000 E. NOTA